Journal of Statistical Physics, Vol. 75, Nos. 1/2, 1994

The Influence of External Boundary Conditions on the Spherical Model of a Ferromagnet. I. Magnetization Profiles

A. E. Patrick[1,2]

Received March 22, 1993; final November 23, 1993

The spherical model of a ferromagnet is investigated for various (external) boundary conditions. It is shown that, besides the well-known critical point, a second one can be produced by the boundary conditions. Although the main asymptotic of the free energy is analytic at the new critical point, the $O(N^{1-2/d})$ asymptotic possesses a singularity here. A natural order parameter of the model has singularities at both critical points. The magnetization profile is studied for the whole range of the model's parameters and at different scales. It is shown that (in an appropriate regime) below the second critical temperature the magnetization profile freezes, that is, becomes temperature independent. Distributions of the single spin variables and some macroscopic observables (including normalized total spin) are studied for the whole temperature range including the critical points.

KEY WORDS: Spherical model; magnetization profile; Gibbs states; phase transitions.

1. INTRODUCTION

At the present time it can be said with confidence that properties of various 2D Ising models with (external) boundary conditions are much better understood than those of similar spherical models (for $d \geqslant 3$) in spite of the fact that investigation of the former needs much more elaborate techniques (for a review see, e.g., ref. 2). Various magnetization profiles for 2D Ising models have been studied by Abraham and Reed[3] and Bariev.[7] Wetting phenomena have been treated in great detail by Abraham,[1] Abraham and

[1] Dublin Institute for Advanced Studies, School of Theoretical Physics, Dublin 4, Ireland.

[2] Present address: Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven, B-3001 Leuven, Belgium.

Issigoni,[5] and Fröhlich and Pfister,[14] to name a few. Various surface effects in the 2D Ising model were examined by Fisher and Ferdinand,[12] McCoy and Wu,[21] and Au Yang,[6] among many others.

Although some of the above phenomena have been treated in spherical models already by Langer,[18] much less is known about these models in comparison with the wealth of results concerning 2D Ising models. Magnetization profiles have been studied by Abraham and Robert,[4] who gave explicit expressions for the profile in the presence of a "global" inhomogeneous external field and for the profile induced by boundary conditions in the high-temperature regime. Surprisingly enough, this paper seems to be the only one where the influence of (external) boundary conditions on the behavior of the spherical model was studied. Some other phenomena induced by inhomogeneous perturbations of the spherical model were considered by Isihara[15] and Barber *et al.*[9]; see also Singh *et al.*[24]

It is generally recognized that investigations of the properties of the original spherical model[10] and its different modifications (see the review by Joyce[16]) were of considerable importance in the formation of our present understanding of critical phenomena and other branches of equilibrium statistical mechanics. The relative simplicity with which the properties of spherical models are analyzed is of great advantage in the investigation of subtle details. Thus, it seems reasonable to undertake a study of the influence of boundary conditions on the behavior of spherical models in analogous situations to those studied for various (undoubtedly, more realistic) 2D Ising models.

In the present paper we propose a systematic approach to study the influence of (external) boundary conditions on the spherical model. Using this approach, we show that the spherical model in the presence of so-called $\pm$-boundary conditions—the model studied earlier by Abraham and Robert[4]—may have two critical points T_c (the critical temperature found by Berlin and Kac[10]) and $\tilde{T}_c$ ($<T_c$). Next we obtain explicit expressions for various magnetization profiles for the model in the whole temperature range; see Eqs. (4.8) and (4.10)–(4.15). As by-products we obtain asymptotic expansions for the free energy up to the order $O(N^{1-2/d})$ (here N is the total number of spins and d is the dimensionality of the model); see Eqs. (3.18), (3.21), and (3.28). This number of terms in the asymptotic expansion is chosen because it is the $O(N^{1-2/d})$ term that has nontrivial behavior at all critical points of the model. An intuitive explanation and interpretation of these features requires knowledge of pair correlation function decay and is postponed till a subsequent paper, where we are going to study the pair correlation functions and "block-spin" variables. In the present paper we also obtain distributions of single spin variables [see Eqs.

(5.3) and (5.7)] and some macroscopic observables in the thermodynamic limit [see Section 5].

The main technical tools we use are a "contour summation" technique (a direct analog of the contour integration in the theory of functions of a complex variable) and rescalings of the integration variable in the integral representations for the partition function, correlation functions, etc., prior to application of the saddle point method. Both tools have been used in a recent paper.[23] It is well known that additional care is needed in applying the saddle point method to the evaluation of the partition function of the spherical model in the low-temperature regime. The problem is that the saddle point in this regime "sticks" at a branch point of the integrand. More precisely, the sequence of saddle points z_N^* (indexed by the number of spins) tends to the branch point as $N \to \infty$. However, if one rescales the integration variable properly, it may happen that this problem disappears, that is, the sequence of (rescaled) saddle points converges to a point of analyticity of the integrand and thus one can apply the saddle point method after the rescaling. Alternatively it may happen that after the rescaling the large parameter N in the integrand "disappears"; then one needs just to investigate the limiting (as $N \to \infty$) integral, which in this case is well defined. As a rule the limiting integral can be evaluated using the contour integration technique. The latter possibility was exploited (apparently) for the first time by Lax.[19]

The paper is organized as follows. The model of interest is described in Section 2. Section 3 is the central part of the paper. There we explain the main technical steps necessary for investigation of the spherical model in the presence of external boundary conditions and derive the asymptotic expansions of the free energy in various regimes. In Section 4 we examine magnetization profiles of the model. Probability distributions of the single spin variables and of some macroscopic observables are obtained in Section 5. Section 6 is devoted to the discussion of the results obtained and final remarks. In Appendix A we explain the "contour summation" technique which is widely used throughout the paper for summation of various sums. In Appendix B convergence rates are found for some sequence of sums which appeared in the main body of the paper.

2. DESCRIPTION OF THE MODEL

The model we study was treated earlier by Abraham and Robert[4] and is defined as follows. Consider a d-dimensional square lattice Z^d for $d \geqslant 3$. There is a spin $\varepsilon_j \in R^1$ at each site $j \equiv (j_1, j_2, \ldots, j_d) \in Z^d$ (in what follows indices without subscript, like j, correspond to d-dimensional vectors, while indices with subscript, like j_ν, correspond to the νth component of a vector j). A system of n^d ($\equiv N$) spins located at sites of a hypercube $\Omega_n = \{j \in Z^d\colon\ 1 \leqslant j_\nu \leqslant n;\ \nu = 1, 2,..., d\}$ interacts via the Hamiltonian

$$H_n(\varepsilon) = -2J \sum_{\langle i, j \rangle} \varepsilon_i \varepsilon_j - \sum_{i \in \Omega_n} h_i \varepsilon_i \tag{2.1}$$

where the first sum runs over all distinct pairs of nearest neighbors. To completely specify the Hamiltonian we must face the question of the boundary conditions (b.c.) to be applied. This involves two steps; first, one needs to specify the nearest neighbors for spins on the border of Ω_n and the interaction between them. We call these boundary conditions internal b.c. Second, one needs to specify the influence of an external environment on the spins in Ω_n (external b.c.). Specification of the internal b.c. modifies the quadratic part of the Hamiltonian and specification of the external b.c. modifies its linear part. We consider internal b.c. which are "empty" in one dimension, say for $\nu = d$, and periodic in all the others, i.e., the set of natural nearest neighbors in Ω_n is complemented by

$$\bigcup_{\nu=1}^{d-1} \{\langle i, j \rangle\colon i_\nu = 1, j_\nu = n,\ i_l = j_l \quad \text{for} \quad l \neq \nu\} \tag{2.2}$$

and the interaction between additional nearest neighbors is the same as for all the other pairs in (2.1). The external b.c. which we consider are specified by the following choice of $\{h_i\}_{i \in \Omega_n}$ in Eq. (2.1):

$$h_{i_1, i_2, ..., i_d} = \begin{cases} h_1 & \text{if} \quad i_d = 1 \\ h_2 & \text{if} \quad i_d = n \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

The joint probability distribution of the random variables $\{\varepsilon_j\}_{j \in \Omega_n}$ (the Gibbs distribution) is defined by the density

$$p(\{\varepsilon_j\}_{j \in \Omega_n}) = \Theta_N^{-1} \exp[-\beta H_n(\{\varepsilon_j\}_{j \in \Omega_n})] \tag{2.4}$$

with respect to the "*a priori*" measure

$$\mu_n(d\varepsilon) = \delta\left(\sum_{j \in \Omega_n} \varepsilon_j^2 - N\right) \prod_{j \in \Omega_n} d\varepsilon_j \tag{2.5}$$

where $\delta(\cdot)$ is the Dirac delta function and $\prod_{j \in \Omega_n} d\varepsilon_j$ is the Lebesgue measure on $(R^N; \mathscr{B}(R^N))$.

For our investigation of the model we need to know the eigenvectors and eigenvalues of the symmetric matrix $\hat{C}^{(n)}$ associated with the quadratic part of the Hamiltonian (2.1)

$$2 \sum_{\langle i,j \rangle} \varepsilon_i \varepsilon_j = \sum_{i;j \in \Omega_n} C^{(n)}_{i;j} \varepsilon_i \varepsilon_j \tag{2.6}$$

where internal b.c. (2.2) are taken into account. These eigenvectors are given by

$$\mathbf{V}_j \equiv \mathbf{V}_{j_1,\dots,j_d} = \left\{ V_{j;m} = \left(\frac{2}{n+1} \right)^{1/2} \sin \frac{\pi j_d m_d}{n+1} \right. \\ \left. \times \prod_{k=1}^{d-1} \frac{1}{\sqrt{n}} \left[\cos \frac{2\pi (j_k - 1)(m_k - 1)}{n} + \sin \frac{2\pi (j_k - 1)(m_k - 1)}{n} \right] \right\}_{m \in \Omega_n} \tag{2.7}$$

and their respective eigenvalues are

$$\lambda_j = 2 \left[\cos \frac{\pi j_d}{n+1} + \sum_{k=1}^{d-1} \cos \frac{2\pi (j_k - 1)}{n} \right] \tag{2.8}$$

3. ASYMPTOTIC EXPANSION FOR THE FREE ENERGY

The partition function of the model is given by

$$\Theta_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[-\beta H_n(\varepsilon)] \, d\mu_n(\varepsilon) \tag{3.1}$$

[see (2.1) and (2.5)]. In the usual fashion, to diagonalize the quadratic part of $H_n(\varepsilon)$ one introduces new integration variables $\{y_s\}_{s \in \Omega_n}$ via

$$\varepsilon_j = \sum_{s \in \Omega_n} V_{j;s} \, y_s \tag{3.2}$$

where the coefficients $V_{j;s}$ are given by (2.7). Next one replaces the delta function in Eq. (3.1) by its integral representation

$$\delta(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\tau x} \, d\tau \tag{3.3}$$

Then integration over $\{y_s\}$ yields

$$\Theta_N = \left(\frac{\pi}{2\beta J} \right)^{N/2} \frac{\beta J}{\pi i} \int_{z_0 - i\infty}^{z_0 + i\infty} dz \, e^{N \Phi_N(z)} \tag{3.4}$$

where

$$\Phi_N(z) = 2\beta Jz - \frac{1}{2} L_d^{(n)}(z) + \frac{\beta}{8JN} \sum_{m \in \Omega_n} \frac{\alpha_m^2}{z - \frac{1}{2}\lambda_m} \tag{3.5}$$

$$L_d^{(n)}(z) = \frac{1}{N} \sum_{j \in \Omega_n} \ln\left(z - \frac{1}{2}\lambda_j\right) \tag{3.6}$$

and the coefficients α_m are given by

$$\alpha_m = \sum_{j_1,\dots,j_d} h_{j_1,\dots,j_d} V_{j_1,\dots,j_d;m_1,\dots,m_d}$$

Below we refer to the transformations yielding (3.4) from (3.1) as "the standard set of transformations." Taking into account Eqs. (2.3), (2.7), one can rewrite α_m as

$$\alpha_m = \left[\frac{2N}{n(n+1)}\right]^{1/2} [h_1 + (-1)^{m_d+1} h_2] \sin\left(\frac{\pi m_d}{n+1}\right) \prod_{k=1}^{d-1} \delta_{m_k;1} \tag{3.7}$$

Note that the presence of external b.c. is summarized by the term

$$T_n(z) \equiv \frac{\beta}{8J} \sum_{j \in \Omega_n} \frac{\alpha_j^2}{z - \frac{1}{2}\lambda_j} \tag{3.8}$$

which we call the field-induced term. Performing the summation in (3.8) (see Appendix A for details), one obtains

$$T_n(z) = n^{-1} N \frac{\beta x_2(z)}{8J} \left[(h_1 - h_2)^2 \frac{x_2^{n-1}(z) - 1}{x_2^{n+1}(z) - 1} + (h_1 + h_2)^2 \frac{x_2^{n-1}(z) + 1}{x_2^{n+1}(z) + 1} \right] \tag{3.9}$$

where

$$x_{1,2}(z) = 1 + z - d \mp [(z-d)(2+z-d)]^{1/2} \tag{3.10}$$

Taking into account Eqs. (3.9) and (B.19) from Appendix B one concludes that for any given $z > d$

$$\Phi_N(z) = 2\beta Jz - \tfrac{1}{2} L_d(z) + n^{-1}\varphi(z) + O(e^{-\gamma n})$$

where

$$\varphi(z) = \frac{1}{4}\left[\frac{\beta x_1(z)}{J}(h_1^2 + h_2^2) + L_{d-1}(z-1) + L_{d-1}(z+1) - 2L_d(z)\right]$$

and

$$L_d(z) = \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{\nu=1}^{d} \frac{d\omega_\nu}{2\pi} \ln\left(z - \sum_{\nu=1}^{d} \cos\omega_\nu\right) = \lim_{N\to\infty} L_d^{(n)}(z) \quad (3.11)$$

Now we evaluate the integral in Eq. (3.4) using the saddle point method. As is usual the relevant saddle point z_n^* is a minimum point of the argument of the exponential in (3.4) on the interval $(\frac{1}{2}\lambda_{1,\dots,1};\infty)$, where $\lambda_{1,\dots,1}$ is the maximal eigenvalue of the matrix $\hat{C}^{(n)}$; see (2.6). Thus, for finite n, z_n^* is a real solution of the equation

$$2\beta J - \tfrac{1}{2} W_d^{(n)}(z) + \frac{1}{N}\frac{d}{dz} T_n(z) = 0 \quad (3.12)$$

satisfying $z_n^* > d - 1 + \cos[\pi/(n+1)]$, where

$$W_d^{(n)}(z) = \tfrac{d}{dz} L_d^{(n)}(z) = \frac{1}{N}\sum_{j\in\Omega_n} \frac{1}{z - \frac{1}{2}\lambda_j} \quad (3.13)$$

The "global" location of the saddle point is governed by the first two terms of the l.h.s. of Eq. (3.12) since the third one is suppressed by n^{-1} [see Eq. (3.9)] and it is described as follows. Let the function

$$\Phi^{(0)}(z) = 2\beta J z - \tfrac{1}{2} L_d(z) \quad (3.14)$$

have a minimum on the interval $[d, \infty)$ at a point z^*. The point z^* is, of course, the limiting saddle point for the integrand in Eq. (3.4) and its location is independent of boundary conditions. According to Berlin and Kac,[10] for any $d \geq 3$ there exists a critical value $\beta_c = W_d(d)/4J < \infty$ [where $W_d(d) \equiv L_d'(d)$] such that $z^* > d$ for $\beta \in [0;\beta_c)$ (the high-temperature regime) and $z^* = d$ for $\beta \in [\beta_c;\infty)$ (the low-temperature regime).

Now, for any $z > d$, the l.h.s. of Eq. (3.12) approaches

$$\phi_n(z) = 2\beta J - \tfrac{1}{2} W_d(z) + n^{-1}\varphi'(z) \quad (3.15)$$

exponentially fast as $N \to \infty$ (see Appendix B) and this rate of convergence is uniform on any interval of the form $\{z\colon a \leq z < \infty,\ a > d\}$. In Eq. (3.15) we used $W_d(z)$ to denote the Watson function

$$W_d(z) = L_d'(z) = \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{\nu=1}^{d} \frac{d\omega_\nu}{2\pi} \frac{1}{z - \sum_{\nu=1}^{d} \cos\omega_\nu} = \lim_{N\to\infty} W_d^{(n)}(z) \quad (3.16)$$

In the high-temperature regime, i.e., when $z^* > d$, Eqs. (3.12) and (3.15) yield

$$z_n^* = z^* + n^{-1}\frac{2\varphi'(z^*)}{W_d'(z^*)} + O(n^{-2}) \tag{3.17}$$

as $N \to \infty$. By applying the saddle point method to the integral in (3.4) one obtains the asymptotic expansion for the free energy, when $\beta \in [0; \beta_c)$,

$$\begin{aligned} F_n(\beta) &= -\frac{1}{\beta}\ln \Theta_N \\ &= N\left[f(\beta; z^*) - N^{-1/d}\frac{\varphi(z^*)}{\beta} - N^{-2/d}\frac{[\varphi'(z^*)]^2}{\beta W_d'(z^*)} + o(N^{-2/d})\right] \end{aligned} \tag{3.18}$$

where

$$f(\beta; z^*) = -\frac{1}{2\beta}\ln\frac{\pi}{2\beta J} - 2Jz^* + \frac{L_d(z^*)}{2\beta} \tag{3.19}$$

is the limiting free emergy per site for the spherical model.

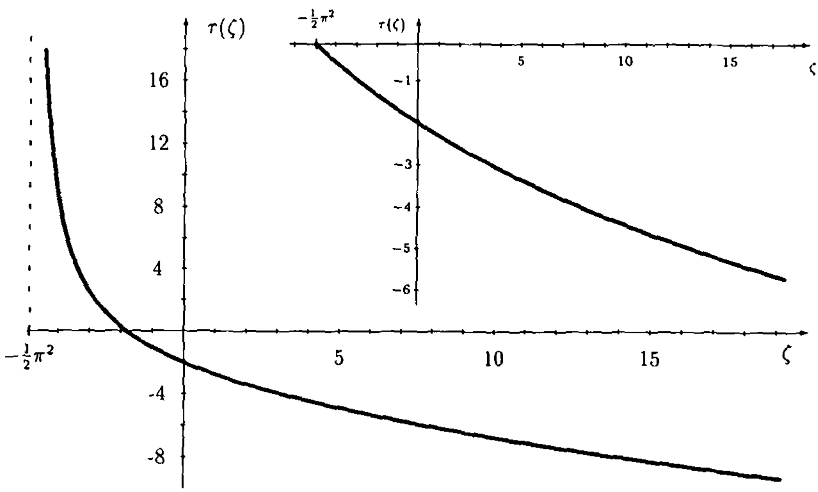

Fig. 1. The function $\tau(\zeta)$ [see Eq. (3.20)] representing the perturbation due to external boundary conditions. The values of the parameters are $(\beta/8J)(h_1 - h_2)^2 = 1$ and $(\beta/8J)(h_1 + h_2)^2 = 1/2$, which means that $h_1 \neq -h_2$ and hence only one critical points exists. In the inset the same function is plotted for $(\beta/8J)(h_1 - h_2)^2 = 1$ and $h_1 = -h_2$. Note that the (right) derivative at the point $-\frac{1}{2}\pi^2$ is finite and hence the second critical point ($\beta = \tilde{\beta}_c$) has a chance of existing.

To be able to apply the saddle point method for the integral in Eq. (3.4) in the low-temperture regime a new integration variable ζ is needed (to pass to a finer scale). We make the ansatz $z = z_n(\zeta) = d + n^{-2}\zeta$. Using Eq. (B.21), we obtain

$$\Theta_N = \left(\frac{\pi}{2\beta J}\right)^{N/2} \frac{\beta J}{n\pi i} \exp\left\{N\left[2\beta J d - \frac{1}{2} L(d) + n^{-1}\varphi(d)\right]\right\}$$
$$\times \int_{\zeta_0 - i\infty}^{\lambda_0 + i\infty} \frac{d\zeta}{(\zeta + \frac{1}{2}\pi^2)^{1/2}}$$
$$\times \exp\left\{N^{1-2/d}\left[2\beta J\zeta - \frac{1}{2} W(d)\zeta + \tau(\zeta)\right] + O(N^{1-3/d})\right\} \quad (3.20)$$

where

$$\tau(\zeta) = -\frac{\beta}{8J}(2\zeta)^{1/2}\left\{(h_1 - h_2)^2 \coth\left[\left(\frac{1}{2}\zeta\right)^{1/2}\right] + (h_1 + h_2)^2 \tanh\left[\left(\frac{1}{2}\zeta\right)^{1/2}\right]\right\}$$

for $\zeta > 0$, and

$$\tau(\zeta) = -\frac{\beta}{8J}(-2\zeta)^{1/2}$$
$$\times\left\{(h_1 - h_2)^2 \cot\left[\left(-\frac{1}{2}\zeta\right)^{1/2}\right] - (h_1 + h_2)^2 \tan\left[\left(-\frac{1}{2}\zeta\right)^{1/2}\right]\right\}$$

for $-\frac{1}{2}\pi^2 < \zeta < 0$; see Fig. 1.

Remark 3.1. Note that the correct analytic continuation of the function

$$f(\zeta) = (2\zeta)^{1/2} \tanh[(\tfrac{1}{2}\zeta)^{1/2}]$$

on the interval $(-\frac{1}{2}\pi^2; 0]$ from the positive semiaxis is given by

$$f(\zeta) = -(-2\zeta)^{1/2} \tan[(-\tfrac{1}{2}\zeta)^{1/2}]$$

as opposed to the naive continuation $f(\zeta) = (-2\zeta)^{1/2} \tan[(-\frac{1}{2}\zeta)^{1/2}]$.

As long as the function

$$\eta(\zeta) = 2J(\beta - \beta_c)\zeta + \tau(\zeta)$$

attains its minimum on the interval $[-\frac{1}{2}\pi^2; \infty)$ at a point $\zeta^* > -\frac{1}{2}\pi^2$ one can apply the saddle point method for the integral in Eq. (3.20). Note that $\zeta = -\frac{1}{2}\pi^2$ is a branch point of the integrand in Eq. (3.20), while $\zeta = 0$ is just a removable singularity. On applying the saddle point method one obtains the following asymptotic expansion for the free energy:

$$F_n(\beta)=N\left\{f(\beta;d)-N^{-1/d}\varphi(d)-N^{-2/d}\left[2J\left(1-\frac{\beta_c}{\beta}\right)\zeta^*+\frac{\tau(\zeta^*)}{\beta}\right]+o(N^{-2/d})\right\} \tag{3.22}$$

where ζ^* is a solution of the equation

$$2J(\beta-\beta_c)+\tau'(\zeta)=0 \tag{3.22}$$

satisfying $\zeta^*>-\frac{1}{2}\pi^2$.

Note that the function $-\tau'(\zeta)$ is monotonically decreasing to zero for $\zeta>-\frac{1}{2}\pi^2$, since $\tau(\zeta)$ was derived from $T_n(z)$ [given by (3.8)] by a uniform change of scale and $-T_n'(z)$ evidently decreases for $z>d-\frac{1}{2}n^{-2}\pi^2$. Consequently, Eq. (3.22) has a unique solution $\zeta^*>-\frac{1}{2}\pi^2$ if $\eta'(-\frac{1}{2}\pi^2)<0$, and the function $\eta(\zeta)$ attains its minimum on $[-\frac{1}{2}\pi^2;\infty)$ at the point $\zeta^*=-\frac{1}{2}\pi^2$ if $\eta'(\zeta)\geqslant 0$. The expansion of $\tau'(\zeta)$ at the point $\zeta=-\frac{1}{2}\pi^2$ is given by

$$\tau'(\zeta)=-\frac{\beta}{8J}\left[\frac{2\pi^2(h_1+h_2)^2}{(\zeta+\frac{1}{2}\pi^2)^2}+\frac{2}{3}(h_1^2-h_1h_2+h_2^2)+\frac{(h_1+h_2)^2}{2\pi^2}\right] + O\left(\zeta+\frac{1}{2}\pi^2\right) \tag{3.23}$$

Hence if $h_1\neq -h_2$ and $\beta>\beta_c$ there always exists a saddle point $\zeta^*>-\frac{1}{2}\pi^2$ for the integrand of (3.20), since for $h_1\neq -h_2$ the derivative of $\tau(\zeta)$ tends to $-\infty$ as $\zeta\to -\frac{1}{2}\pi^2$. When $h_1=-h_2=h$ the right derivative of $\tau(\zeta)$ at the point $-\frac{1}{2}\pi^2$ is equal to $-(\beta/4J)h^2$ and a saddle point $\zeta^*>-\frac{1}{2}\pi^2$ exists for all $\beta>\beta_c$ only if $\frac{1}{2}(h/2J)^2\geqslant 1$. If, however, $\frac{1}{2}(h/2J)^2<1$, then a saddle point $\zeta^*>-\frac{1}{2}\pi^2$ exists only for $\beta\in(\beta_c;\tilde{\beta}_c)$, where

$$\tilde{\beta}_c=\frac{\beta_c}{1-\frac{1}{2}(h/2J)^2}$$

Thus $\tilde{\beta}_c$ (as well as β_c) is a "sticking" point and we will see later that it deserves to be called a critical temperature for the spherical model.

Since for $\beta>\tilde{\beta}_c$ the saddle point ζ^* "sticks" at the branch point $\zeta=-\frac{1}{2}\pi^2$, one needs to again rescale the integration variable in (3.20) via $\tilde{\zeta}=(\zeta+\frac{1}{2}\pi^2)N^{1-2/d}$, which together with Eqs. (B.22) and (B.23) yields

$$\Theta_N=\left(\frac{\pi}{2\beta J}\right)^{N/2}\exp\left\{N\left[2\beta Jd-\frac{1}{2}L(d)+n^{-1}\varphi(d)-n^{-2}J\pi^2(\beta-\beta_c)+o(n^{-2})\right]\right\}$$
$$\times\frac{\beta J}{\sqrt{N}\pi i}\int_{\tilde{\zeta}_0-i\infty}^{\tilde{\zeta}_0+i\infty}\frac{d\tilde{\zeta}}{(\tilde{\zeta})^{1/2}}\left\{\tilde{\zeta}\left[2\beta J-\frac{1}{2}W(d)-\frac{\beta h^2}{4J}\right]+o(1)\right\} \tag{3.24}$$

where the integral makes a contribution only to the $O(1)$ terms of the asymptotic expansion for the free energy.

Summarizing the above, one concludes that the asymptotic expansion for $F_n(\beta)$ in the low-temperature regime is given by Eqs. (3.21) and (3.22), and if $h_1=-h_2=h$, $\frac{1}{2}(h/2J)^2<1$, and $\beta>\tilde{\beta}_c$, one sets $\zeta^*=-\frac{1}{2}\pi^2$.

To complete the whole picture one should calculate the asymptotic expansion of $F_n(\beta)$ at $\beta=\beta_c$ [the corresponding expansion at $\tilde{\beta}_c$ can be obtained by setting $\beta\to\tilde{\beta}_c$ in Eq. (3.21)]. As in the case $\beta>\beta_c$, it is necessary to rescale the integration variable z in Eq. (3.4) to obtain an integral for which the saddle point method can be applied. The way one needs to rescale z depends on dimension, since the expansion of the function $L_d(z)$ near $z=d$ possesses a nontrivial dependence on dimension,[8]

$$L_d(z)=L_d(d)+W_d(d)(z-d)$$
$$+\begin{cases}-\dfrac{\sqrt{2}}{3\pi}(z-d)^{3/2}+O[(z-d)^2] & \text{if } d=3\\[2mm] \dfrac{1}{8\pi^2}(z-d)^2\ln(z-d)+O[(z-d)^2] & \text{if } d=4\\[2mm] \dfrac{1}{2}W_d'(d)(z-d)^2+o[(z-d)^2] & \text{if } d\geqslant 5\end{cases} \tag{3.25}$$

After the rescaling

$$z=z_n(\zeta)=d+v_d(n)\zeta,\quad \text{where } v_d(n)=\begin{cases}n^{-1} & \text{for } d=3\\ (n\ln n)^{-2/3} & \text{for } d=4\\ n^{-2/3} & \text{for } d\geqslant 5\end{cases} \tag{3.26}$$

we are able to apply the saddle point method to the evaluation of the integral in Eq. (3.4). The location of the saddle point is given by

$$\zeta^*=\begin{cases}\dfrac{\pi\beta(h_1^2+h_2^2)}{2J} & \text{for } d=3\\[2mm] \left[\dfrac{3\pi^2\beta(h_1^2+h_2^2)}{2\sqrt{2}J}\right]^{2/3} & \text{for } d=4\\[2mm] \left[-\dfrac{\beta(h_1^2+h_2^2)}{2\sqrt{2}JW_d'(d)}\right]^{2/3} & \text{for } d\geqslant 5\end{cases} \tag{3.27}$$

and the asymptotic expansion of the free energy is

$$F_n(\beta_c) = Nf(\beta_c; d) - N^{1-1/d}\frac{\varphi(d)}{\beta} + \begin{cases} N^{1/2}c_3 + o(N^{1/2}) & \text{if } d=3 \\ (\ln N)^{1/3} N^{2/3} c_4 + O(N^{2/3}) & \text{if } d=4 \\ N^{1-4/(3d)} c_d + o(N^{1-4/(3d)}) & \text{if } d \geqslant 5 \end{cases} \tag{3.28}$$

The amplitudes $\{c_d\}_{d\geqslant 3}$ in the last expression can be calculated exactly; for instance, when $d=3$, one obtains

$$c_3 = \frac{(\pi\beta_c)^{1/2}}{3}\left(\frac{h_1^2 + h_2^2}{2J}\right)^{3/2}$$

To conclude the section, let us stress a curious point. Above we found that for the model under consideration there is a range of parameters $[h_1 = -h_2 = h,\ \frac{1}{2}(h/2J)^2 < 1]$ for which the amplitude $f_2(\beta; d)$ in the asymptotic expansion of the free energy

$$F_n(\beta) = Nf(\beta; d) + N^{1-1/d} f_1(\beta; d) + N^{1-2/d} f_2(\beta; d) + \cdots$$

has two points of nonanalyticity when considered as a function of β. The amplitude $f_2(\beta; d)$ is continuous at one of them $(\tilde{\beta}_c)$ and goes to infinity at the other (β_c). Summarizing Eqs. (3.18) and (3.21), one obtains the following asymptotic for $f_2(\beta; d)$

$$f_2(\beta; d) = \frac{(h_1^2 + h_2^2)^2}{(4J)^3}\left|1 - \frac{\beta_c}{\beta}\right|^{-1} + o\left(\left|1 - \frac{\beta_c}{\beta}\right|^{-1}\right)$$

as $\beta \to \beta_c$. The magnitude to which $f_2(\beta; d)$ actually "blows up" for the finite system at the (now pseudo) critical point β_c depends on dimension and, according to (3.28), is given by

$$f_2^{(n)}(\beta_c; d) = \begin{cases} \sqrt{n}\, c_3 + o(\sqrt{n}) & \text{if } d=3 \\ (\ln n)^{1/3} n^{2/3} c_4 + O(n^{2/3}) & \text{if } d=4 \\ n^{2/3} c_d + o(n^{2/3}) & \text{if } d \geqslant 5 \end{cases} \tag{3.29}$$

4. MAGNETIZATION PROFILE

The average value of a single spin variable ε_k is defined by

$$\langle \varepsilon_k \rangle_N = \Theta_N^{-1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varepsilon_k \exp[-\beta H_n(\varepsilon)]\, d\mu_n(\varepsilon) \tag{4.1}$$

After the standard set of transformations one obtains the following formula for the average value:

$$\langle \varepsilon_k \rangle_N = \frac{1}{4J} \left\langle \sum_{s \in \Omega_n} \frac{V_{k;s} \alpha_s}{z - \frac{1}{2}\lambda_s} \right\rangle_{z,N}, \qquad k \in \Omega_n \tag{4.2}$$

where the notation

$$\langle f(z) \rangle_{z,N} = \Theta_N^{-1} \left(\frac{\pi}{2\beta J} \right)^{N/2} \frac{\beta J}{\pi i} \int_{z_0 - i\infty}^{z_0 + i\infty} dz\, f(z)\, e^{N\Phi_N(z)} \tag{4.3}$$

was introduced. It is possible to calculate the sums

$$\Sigma_k(z) \equiv \sum_{s \in \Omega_n} \frac{V_{k;s} \alpha_s}{z - \frac{1}{2}\lambda_s} \tag{4.4}$$

exactly (see Appendix A) with the result

$$\Sigma_k(z) = (h_1 - h_2) \left[\frac{x_2^{n+1-k_d}(z) - x_2^{k_d}(z)}{x_2^{n+1}(z) - 1} \right] + (h_1 + h_2) \left[\frac{x_2^{n+1-k_d}(z) + x_2^{k_d}(z)}{x_2^{n+1}(z) + 1} \right] \tag{4.5}$$

where $x_{1,2}(z)$ is given by (3.10). Since $\langle \varepsilon_k \rangle_N$ is just the average (4.3) of $\Sigma_k(z)$, it follows from Eq. (4.5) that $\langle \varepsilon_k \rangle_N$ depends only on k_d and hence it is sufficient to study just one row, say $(1, \ldots, 1, k_d)$, of the original hypercube Ω_n. To be able to consider the problem in a variety of scales in the thermodynamic limit we introduce the notation

$$\langle \varepsilon_\gamma \rangle_{\nu;\delta} \equiv \lim_{N \to \infty} \langle \varepsilon_{1,\ldots,1,[n^\delta \gamma + \nu n]} \rangle_N \tag{4.6}$$

where $[n^\delta \gamma + \nu n]$ stands for the integer part of $n^\delta \gamma + \nu n$, $0 \leqslant \delta \leqslant 1$, and $0 \leqslant \nu \leqslant 1$. Note that γ is a continuous variable unless $\delta = 0$ (no rescaling), and its range can be easily established; for instance, $\gamma \in (-\infty; \infty)$ if $0 < \delta < 1$, $\nu \neq 0, 1$, and $\gamma \in [0, 1]$ if $\delta = 1$, $\nu = 0$.

Unless $h_1 = -h_2$, $\frac{1}{2}(h_1/2J)^2 < 1$, and $\beta > \tilde{\beta}_c$ one can apply the saddle point method for evaluation of the integral in Eq. (4.2) (possibly after prior rescaling of the integration variable) with the result

$$\langle \varepsilon_\gamma \rangle_{\nu;\delta} = \lim_{N \to \infty} \frac{1}{4J} \Sigma_{1,\ldots,1,[n^\delta \gamma + \nu n]}(z_n^*) \tag{4.7}$$

where z_n^* is the sequence of saddle points.

In the high-temperature regime, i.e., for $\beta<\beta_c$, the sequence z_n^* converges to $z^*>d$ [cf. (3.17)]; hence

$$\begin{aligned}\langle\varepsilon_k\rangle_{0;0}&=\frac{h_1}{2J}x_1^k(z^*), \qquad k=1,2,\ldots\\ \langle\varepsilon_k\rangle_{1;0}&=\frac{h_2}{2J}x_2^k(z^*), \qquad k=-1,-2,\ldots\end{aligned} \tag{4.8}$$

and $\langle\varepsilon_k\rangle_{\nu;0}=0$ for $\nu\neq 0, 1$. Thus, in the high-temperature regime the average values decay exponentially fast as we move from the surface into the bulk. Equation (4.8) was derived earlier by Abraham and Robert,[4] who obtained it for the particular case $h_1=h$, $h_2=0$.

At the critical point $\beta=\beta_c$ the location of the saddle point scales with n according to Eq. (3.26), and hence one should choose the rescaling exponent in Eq. (4.6) as follows: $\delta=\frac{1}{2}$ for $d=3$ and $\delta=\frac{1}{3}$ for $d\geqslant 5$; for $d=4$, however, a rescaled cordinate γ has to be introduced by

$$\langle\varepsilon_\gamma\rangle_{\nu;\delta}^{(d=4)}\equiv\lim_{N\to\infty}\langle\varepsilon_{1,\ldots,1,[(n\ln n)^\delta\gamma+\nu n]}\rangle_N \tag{4.9}$$

with $\delta=\frac{1}{3}$. In these scales one obtains

$$\begin{aligned}\langle\varepsilon_\gamma\rangle_{0;\delta}&=\frac{h_1}{2J}\exp[-\gamma(2\zeta^*)^{1/2}] \qquad \gamma\in[0;\infty)\\ \langle\varepsilon_\gamma\rangle_{1;\delta}&=\frac{h_2}{2J}\exp[\gamma(2\zeta^*)^{1/2}] \qquad \gamma\in(-\infty;0]\end{aligned} \tag{4.10}$$

where ζ^* is given by (3.27). Note that for $\beta=\beta_c$ unphysical properties of the spherical model already start to emerge. These are related to the infinite-range interaction (due to the spherical constraint). Indeed, the magnetization profile near the left (right) boundary depends on the magnitude of the external field $h_2(h_1)$ near the right (left) boundary, in spite of the fact that it decays very fast ($\delta<1$); see (3.27).

For $h_1\neq -h_2$ and $\beta>\beta_c$ the location of the saddle point scales as $z_n(\zeta^*)=d+n^{-2}\zeta^*$, where ζ^* is the solution of the equation given in (3.22) satisfying $\zeta^*>-\frac{1}{2}\pi^2$. Hence Eqs. (4.5) and (4.7) in this case yield

$$\langle\varepsilon_\gamma\rangle_{0;1}=\frac{h_1-h_2}{4J}\frac{\sinh[(1-2\gamma))(\frac{1}{2}\zeta^*)^{1/2}]}{\sinh[(\frac{1}{2}\zeta^*)^{1/2}]}+\frac{h_1+h_2}{4J}\frac{\cosh[(1-2\gamma)(\frac{1}{2}\zeta^*)^{1/2}]}{\cosh[(\frac{1}{2}\zeta^*)^{1/2}]} \tag{4.11}$$

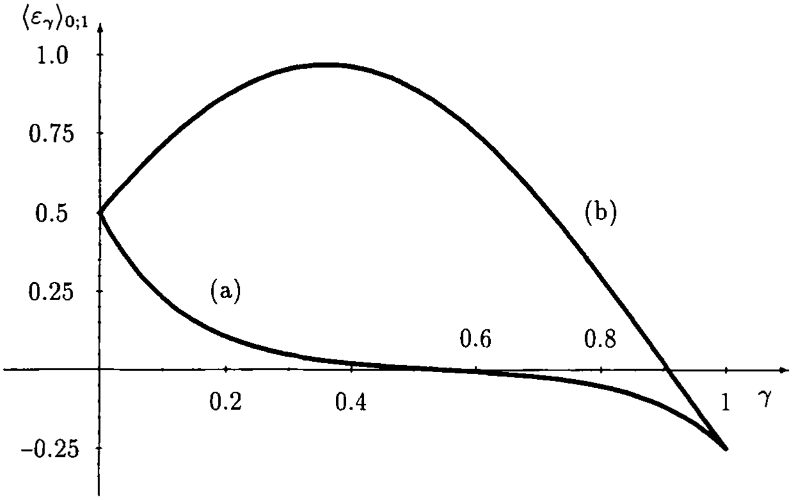

Fig. 2. The magnetization profiles given by Eqs. (4.11), (4.12) for $h_1 = 1$, $h_2 = -\frac{1}{2}$, $J = 1$, and (a) $1-\beta_c/\beta = 0.02$ or (b) $1-\beta_c/\beta = 0.5$. The exchange interaction for these values of J, h_1, and h_2 is strong enough to make the low-temperature magnetization profile similar to the ground-state configuration of the system with $h_{1,2} = 0$.

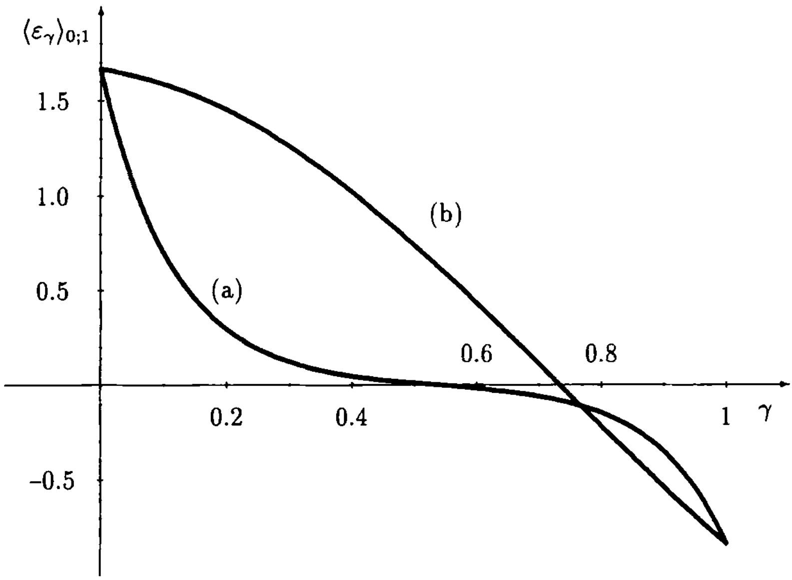

Fig. 3. The magnetization profiles given by Eqs. (4.11), (4.12) for $h_1 = 1$, $h_2 = -\frac{1}{2}$, $J = 0.3$, and (a) $1-\beta_c/\beta = 0.2$ or (b) $1-\beta_c/\beta = 1$. The external boundary conditions for these values of J, h_1, and h_2 prevent the magnetization profile from looking similar to the ground-state configuration of the unperturbed system (with $h_{1,2} = 0$) even at zero temperature.

when $\zeta^* \geqslant 0$, and

$$\langle \varepsilon_\gamma \rangle_{0;1} = \frac{h_1 - h_2}{4J} \frac{\sin[(1-2\gamma))(-\frac{1}{2}\zeta^*)^{1/2}]}{\sin[(-\frac{1}{2}\zeta^*)^{1/2}]} + \frac{h_1 + h_2}{4J} \frac{\cos[(1-2\gamma)(-\frac{1}{2}\zeta^*)^{1/2}]}{\cos[(-\frac{1}{2}\zeta^*)^{1/2}]} \tag{4.12}$$

when $\zeta^* \geqslant 0$ (see Figs. 2 and 3).

Finally, for $h_1 = -h_2 = h$ and $\beta_c \leqslant \beta \leqslant \tilde{\beta}_c$ the magnetization profile is given by

$$\langle \varepsilon_\gamma \rangle_{0;1} = \frac{h}{2J} \frac{\sinh[(1-2\gamma)(\frac{1}{2}\zeta^*)^{1/2}]}{\sinh[(\frac{1}{2}\zeta^*)^{1/2}]}, \qquad \gamma \in [0, 1] \tag{4.13}$$

and

$$\langle \varepsilon_\gamma \rangle_{0;1} = \frac{h}{2J} \frac{\sin[(1-2\gamma)(-\frac{1}{2}\zeta^*)^{1/2}]}{\sin[(-\frac{1}{2}\zeta^*)^{1/2}]}, \qquad \gamma \in [0, 1] \tag{4.14}$$

when $\zeta^* \geqslant 0$ and $\zeta^* \leqslant 0$, respectively. When $\beta > \tilde{\beta}_c$ the magnetization profile is given by

$$\langle \varepsilon_\gamma \rangle_{0;1} = \frac{h}{2J} \sin\left[\left(\frac{1}{2} - \gamma\right)\pi\right], \qquad \gamma \in [0, 1] \tag{4.15}$$

since in the scale $d + n^{-2}\zeta$ the saddle point sticks at the point $\zeta^* = -\frac{1}{2}\pi^2$. Thus, for $h_1 = -h_2$ and $\beta > \tilde{\beta}_c$ the magnetization profile freezes, i.e., becomes temperature independent.

5. DISTRIBUTIONS OF SPINS AND OF SOME MACROSCOPIC OBSERVABLES

In this section we derive formulas for distributions of single spin variables, of the properly normalized total spin, and of the thermodynamic variable

$$p_\gamma \stackrel{d}{=} \lim_{N \to \infty} \frac{1}{N^\gamma} \sum_{j \in \Omega_n} \delta_j \varepsilon_j \tag{5.1}$$

where $\delta_{j_1,\dots,j_d} = \operatorname{sgn}(j_d - \frac{1}{2}n)$ and the value of γ will be specified later. The distribution for the variable p_γ will be calculated only in the case $h_1 = -h_2$ since only then does it produce meaningful information. All the distributions can be obtained using the method of characteristic functions. In many cases we arrive at characteristic functions of the form

$$\chi(t) = \exp(iat - \tfrac{1}{2}b^2t^2)$$

which corresponds, of course, to a Gaussian distribution with mean a and variance b. Since the calculations are essentially the same as those in the previous sections we present them very briefly.

For the characteristic function $\langle \exp(it\varepsilon_k)\rangle_N$ of a spin variable ε_k [cf. (4.1)] one obtains after the standard set of transformations

$$\langle \exp(it\varepsilon_k)\rangle_N = \left\langle \exp\left[\frac{it}{4J}\Sigma_k(z) - \frac{t^2}{8\beta J}D_k(z)\right]\right\rangle_{z,N} \tag{5.2}$$

where

$$D_k(z) = \sum_{j\in\Omega_n} \frac{V_{k;j}^2}{z-\frac{1}{2}\lambda_j}$$

and $\Sigma_k(z)$ is given by (4.4). As long as one can evaluate the integral on the r.h.s. of (5.2) using the saddle point method the limiting $(N\to\infty)$ characteristic function [in the notation of Eq. (4.6)] is given by

$$\langle \exp(it\varepsilon_\gamma)\rangle_{\nu;\delta} = \exp\left(it\langle\varepsilon_\gamma\rangle_{\nu;\delta} - \frac{t^2}{8\beta J}\langle d_\gamma\rangle_{\nu;\delta}\right) \tag{5.3}$$

where

$$\langle d_\gamma\rangle_{\nu;\delta} \equiv \lim_{N\to\infty} D_{1,\dots,1,[n^\delta\gamma+\nu n]}(z^*)$$

z^* is the limiting saddle point, and, depending on the model's parameter values, $\langle\varepsilon_\gamma\rangle_{\nu;\delta}$ is given by one of Eqs. (4.8)–(4.12).

In the high-temperature regime $(\beta<\beta_c)$, $\langle d_\gamma\rangle_{0;0}$ is given by

$$\langle d_k\rangle_{0;0} = \frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int_0^{2\pi} d\omega_1\cdots d\omega_d \frac{1-\cos(2\omega_d k_d)}{z^*-\sum_{l=1}^d \cos\omega_l}, \quad k_d = 1, 2,\dots \tag{5.4}$$

The same expression is valid for $\langle d_k\rangle_{1;0}$ except that k_d now runs on the set of negative integers. For all other values of $\nu\,;\delta$

$$\langle d_\gamma\rangle_{\nu;\delta} = W_d(z^*)$$

Note that $W_d(z^*) = 4\beta J$ for $\beta\leqslant\beta_c$. In the temperature interval $\beta_c\leqslant\beta\leqslant\tilde{\beta}_c$ one just has to substitute $z^*=d$ in the above formulas for $\langle d_k\rangle_{\nu;\delta}$.

The sequence $\langle d_k\rangle_{0;0}$ steadily increases toward its bulk value $W_d(z^*)$ as $k_d\to\infty$, and the following relations are valid:

$$W_d(z^*) - \langle d_k\rangle_{0;0} \sim x_1^{k_d}(z^*)\, 2^{2-d}(\pi k_d)^{(1-d)/2}[x_2(z^*)-x_1(z^*)]^{(d-3)/2}$$

when $z^* > d$, that is, in the high-temperature regime, and

$$W_d(z^*) - \langle d_k \rangle_{0;0} \sim a_d k_d^{2-d} \tag{5.5}$$

when $z^* = d$, i.e.,. for $\beta \geqslant \beta_c$ ($b_k \sim c_k$ means, as usual, that $\lim_{k\to\infty} b_k/c_k = 1$). The amplitudes a_d depend only on dimension and can be calculated exactly; for instance, $a_3 = 1/(2\pi)$.

To evaluate the integral in Eq. (5.2) for $h_1 = -h_2 = h$, $\frac{1}{2}(h/2J)^2 \leqslant 1$, and $\beta > \tilde{\beta}_c$ one needs to introduce a new integration variable ζ via $z = \lambda_{1,\ldots,1} + N^{-1}\zeta$, which yields $\langle \exp(it\varepsilon_\gamma)\rangle_{0;1} = I_1/I_2$, where

$$I_1 = \exp\left[it\langle \varepsilon_\gamma \rangle_{0;1} - \frac{t^2}{8\beta J}\langle d_\gamma \rangle_{0;1}\right] \int_{\zeta_0 - i\infty}^{\zeta_0 + i\infty} \frac{d\zeta}{\sqrt{\zeta}} \exp\left(2\zeta\beta J m_s^2 - \frac{t^2 \sin^2 \pi\gamma}{4\zeta\beta J}\right)$$

$$I_2 = \int_{\zeta_0 - i\infty}^{\zeta_0 + i\infty} \frac{d\zeta}{\sqrt{\zeta}} \exp(2\zeta\beta J m_s^2)$$

and

$$m_s^2 = 1 - \frac{\beta_c}{\beta} - \frac{1}{2}\left(\frac{h}{2J}\right)^2 \tag{5.6}$$

Thus, the characteristic function $\langle \exp(it\varepsilon_\gamma)\rangle_{0;1}$ for $\beta > \tilde{\beta}_c$ is given by[25]

$$\langle \exp(it\varepsilon_\gamma)\rangle_{0;1} = \exp\left(it\langle \varepsilon_\gamma \rangle_{0;1} - \frac{t^2}{8\beta J}\langle d_\gamma \rangle_{0;1}\right) \cos[tm_s\sqrt{2}\sin(\pi\gamma)]$$

and the corresponding distribution density is

$$p_\gamma(\varepsilon) = \frac{1}{2}\left(\frac{2\beta J}{\pi\langle d_\gamma \rangle_{0;1}}\right)^{1/2} \times \sum_{\sigma=\pm 1} \exp\left\{-\frac{2\beta J[\varepsilon - (h/2J)\sin\pi(\frac{1}{2}-\gamma) - \sigma m_s\sqrt{2}\sin\pi\gamma]^2}{\langle d_\gamma \rangle_{0;1}}\right\} \tag{5.7}$$

Similar calculations for the characteristic function of the normalized total spin yield

$$\left\langle \exp\left(it\sum_{j\in\Omega_n} \varepsilon_j\right)\right\rangle_N = \langle \exp[it\sigma_1(z) - \tfrac{1}{2}t^2\sigma_2(z)]\rangle_{z,N} \tag{5.8}$$

where $\sigma_1(z)$ and $\sigma_2(z)$ are given by

$$\sigma_1(z) = \frac{1}{4J}\sum_{m\in\Omega_n} \frac{\alpha_m\gamma_m}{z - \frac{1}{2}\lambda_m} \tag{5.9}$$

$$\sigma_2(z) = \frac{1}{4\beta J}\sum_{m\in\Omega_n} \frac{\gamma_m^2}{z - \frac{1}{2}\lambda_m} \tag{5.10}$$

The summations in Eqs. (5.9), (5.10) can be performed (see Appendix A) with the results

$$\sigma_1(z)=\frac{N}{n}\frac{h_1+h_2}{4J}\left[\frac{2(2+z-d)}{x_2(z)-x_1(z)}\frac{x_2^{n+1}(z)-1}{x_2^{n+1}(z)+1}-1\right]$$

$$\sigma_2(z)=\frac{N}{4\beta J(z-d)}\left\{1+\frac{1}{n}-\frac{2(2+z-d)}{n[x_2(z)-x_1(z)]}\frac{x_2^{n+1}(z)-1}{x_2^{n+1}(z)+1}\right\}$$

Hence in the high-temperature regime one obtains

$$\lim_{N\to\infty}\left\langle \exp\left(itN^{1/d-1}\sum_{j\in\Omega_n}\varepsilon_j\right)\right\rangle_N=\exp\left\{it\frac{h_1+h_2}{4J}\left[\left(1+\frac{2}{z^*-d}\right)^{1/2}-1\right]\right\} \tag{5.11}$$

Let us now consider the characteristic function

$$\chi_N(t)\equiv\left\langle \exp\left[it\sum_{j\in\Omega_n}\varepsilon_j-it\sigma_1(z_n^*)\right]\right\rangle_N \tag{5.12}$$

of total spin fluctuations around $\sigma_1(z_n^*)$, where z_n^* is given by Eq. (3.17). According to Eq. (5.8),

$$\chi_N(t)=\langle \exp\{it[\sigma_1(z)-\sigma_1(z_n^*)]-\tfrac{1}{2}t^2\sigma_2(z)\}\rangle_{z,N} \tag{5.13}$$

Since we subtracted the mean value from the total spin, we have to compensate for the growth of $\sigma_2(z)$ to obtain a nontrivial characteristic function in the limit $N\to\infty$. Thus we introduce the normalization factor $\sqrt{N}$ and study the sequence $\chi_N(N^{-1/2}t)$. The term linear in t in Eq. (5.13) is of the order $n^{-1}N^{1/2}$ after the normalization and hence produces some perturbation to the saddle point equation. However, σ_1 is analytic in a small enough neighborhood of the point $z^*>d$ and, being suppressed by $n^{-1}N^{-1/2}$ in the saddle point equation, leads only to a "slight" deformation of the saddle surface and to a deviation in the saddle point location of the order $n^{-1}N^{-1/2}$. Consequently,

$$\lim_{N\to\infty}\chi_N(t)=\exp\left[-\frac{t^2}{8\beta J(z^*-d)}\right]$$

Thus, for $\beta<\beta_c$ the fluctuations of the normalized total spin have a standard Gaussian distribution and, as could have been anticipated, the magnitude of the total spin's "nonrandom" part is proportional to the number of spins on the surface of Ω_n.

At the first critical temperature one needs to rescale the integration variable in (5.8) according to Eq. (3.26). Then application of the saddle point method yields

$$\lim_{N\to\infty}\left\langle \exp\left[itg_d^{-1}(n)\sum_{j\in\Omega_n}\varepsilon_j\right]\right\rangle_N = \exp\left[it\frac{h_1+h_2}{4J}\left(\frac{2}{\zeta^*}\right)^{1/2}\right]$$

where

$$g_d(n)=\begin{cases} n^{d-1/2} & \text{if } d=3\\ (\ln n)^{1/3}n^{d-2/3} & \text{if } d=4\\ n^{d-2/3} & \text{if } d\geqslant 5\end{cases} \tag{5.14}$$

and ζ^* is given by Eq. (3.27).

After rescaling (3.26) the characteristic function $\chi_N(t)$ can be written as

$$\chi_N(t)=\left\langle \exp\left[itn^{-1}N[2v_d^{-1}(n)]^{1/2}\frac{h_1+h_2}{4J}\left(\frac{1}{\sqrt{\zeta}}-\frac{1}{\sqrt{\zeta_n^*}}\right)\right.\right.$$
$$\left.\left. -Nv_d^{-1}(n)\frac{t^2}{8\beta J\zeta}\right]\right\rangle_{z_n(\zeta),N}$$

where $v_d(n)$ is given by (3.26) and insignificant correction terms were omitted. To obtain a nontrivial limit for $\chi_N(t)$ one has to compensate for the growth of the term quadratic in t via the introduction of the rescaling factor $[N^{-1}v_d(n)]^{1/2}$ for t. The term linear in t is analytic in a neighborhood of the point ζ_n^* and thus produces only a saddle point shift of the order $n^{-1}N^{-1/2}v_d^{-3/2}(n)$. Consequently, at the point $\beta=\beta_c$

$$\lim_{N\to\infty}\chi_N([N^{-1}v_d(n)]^{1/2}t)=\exp\left(-\frac{t^2}{8\beta J\zeta^*}\right) \tag{5.15}$$

where $z_n(\zeta^*)$ is the saddle point given by Eqs. (3.26), (3.27). Note that since the phase transition at T_c is not accompanied by spontaneous symmetry breaking the distribution of the normalized total spin remains Gaussian at the critical point; however, the normalizing factor becomes abnormally large.

In the low-temperature regime for $\beta_c<\beta<\tilde{\beta}_c$ one needs to rescale the integration variable in Eq. (5.8) via $z=d+\zeta n^{-2}$. On rescaling one can apply the saddle point method, obtaining

$$\lim_{N\to\infty}\left\langle \exp\left(it\frac{1}{N}\sum_{j\in\Omega_n}\varepsilon_j\right)\right\rangle_N = \exp\left\{it\frac{h_1+h_2}{4J}\left(\frac{2}{\zeta^*}\right)^{1/2}\tanh\left[\left(\frac{\zeta^*}{2}\right)^{1/2}\right]\right\} \tag{5.16}$$

where one has to replace the r.h.s. by its analytic continuation (from the positive semiaxes) for $-\frac{1}{2}\pi^2 < \zeta^* \leqslant 0$.

One can consider the characteristic function $\chi_N(t)$ for $\beta_c < \beta < \tilde{\beta}_c$ in much the same way as was done for $\beta = \beta_c$. The proper rescaling factor in this case is $N^{-1/2-1/d}$. Thus one needs to consider the sequence $\chi_N(N^{-1/2-1/d}t)$. Taking into account the perturbation produced by the linear term in t, one concludes that the deviation of the saddle point is given by

$$\zeta_n(t) = \zeta_n^* - itnN^{-1/2}\frac{(h_1+h_2)\,s'(\zeta_n^*)}{4J\tau''(\zeta_n^*)} + o(N^{-1/2}n)$$

where $s(\zeta) = (2/\zeta)^{1/2}\tanh[(\zeta/2)^{1/2}]$ and $\tau(\zeta)$ was introduced in Eq. (3.20). Contrary to the case $\beta \leqslant \beta_c$, the deviation of the saddle point makes a nonvanishing contribution to the variance of total spin fluctuations. Indeed, having applied the saddle point method for the evaluation of $\chi_N(N^{-1/2-1/d}t)$, one arrives at

$$\lim_{N\to\infty} \chi_N(N^{-1/2-1/d}t) = \exp\left(-\frac{t^2}{4\beta J\zeta^*}\left\{1-\left(\frac{2}{\zeta^*}\right)^{1/2}\tanh\left[\left(\frac{\zeta^*}{2}\right)^{1/2}\right]\right\} + t^2\left(\frac{h_1+h_2}{4J}\right)^2\frac{[s'(\zeta^*)]^2}{2\tau''(\zeta^*)}\right) \tag{5.17}$$

and the same analytic continuation as in Eq. (5.16) has to be done for negative values of ζ^*.

At the second critical point $\beta = \tilde{\beta}_c$, which is accompanied by symmetry breaking, one has a pretty interesting situation. For this temperature one needs to rescale the integration variable in Eq. (5.8) according to

$$z = d - \tfrac{1}{2}\pi^2 N^{-2/d} + \zeta N^{-1/d-1/2} \tag{5.18}$$

which yields (as $N \to \infty$)

$$\left\langle \exp\left(it\sum_{j\in\Omega_n}\varepsilon_j\right)\right\rangle_N = \frac{I_1}{I_2}$$

where

$$I_1 = \int_{\zeta_0-i\infty}^{\zeta_0+i\infty}\frac{1}{\sqrt{\zeta}}\exp\left\{\frac{\beta h^2\zeta^2}{8J\pi^2} + o(1) - \frac{t^2N^{1+2/d}}{\beta J\pi^2\zeta}\,[N^{1/2-1/d} + O(1)]\right\} \tag{5.19}$$

and

$$I_2 = \int_{\zeta_0-i\infty}^{\zeta_0+i\infty}\frac{1}{\sqrt{\zeta}}\exp\left[\frac{\beta h^2\zeta^2}{8J\pi^2} + o(1)\right] \tag{5.20}$$

One sees from Eq. (5.19) that one needs to choose the normalizing factor $N^{-3/4-1/(2d)}$ for the total spin in order that its limiting (as $N \to \infty$) distribution be a nontrivial one. Note that all points of nonanalyticity of the 'o' terms in Eqs. (5.19), (5.20) move to infinity along the negative semiaxis as $N \to \infty$. Indeed, these terms originate in integrals of the type (3.4) and all singularities of such integrals, except for the branch point corresponding to $(z-\lambda_{1,\dots,1})^{1/2}$, move to $z=\infty$ along the negative semiaxis under the rescaling (5.18). Obviously $(z-\lambda_{1,\dots,1})^{1/2}$ is not included in the 'o' terms. Hence closing the contour around a keyhole on the negative semiaxis contains no singularities and our original contour can be replaced by the one over the negative semiaxis. On integrating and performing the inverse Fourier transform of the characteristic function, one arrives at the distribution function

$$\Pr\left[N^{-3/4-1/(2d)} \sum_{j\in\Omega_n} \varepsilon_j \leqslant x\right] = \frac{2}{\Gamma(1/4)}\left(\frac{2\tilde{\beta}_c J^3\pi^6}{h^2}\right)^{1/4} \int_{-\infty}^{x} dy \exp\left(-\frac{2\tilde{\beta}_c J^3\pi^6}{h^2} y^4\right)$$

Thus, at the symmetry-breaking critical point the presence of external boundary conditions "washes out" the dependence on dimension of the (properly normalized) total spin distribution, which the spherical model without external b.c. enjoys.[22] The normalization factor $N^{-3/4-1/(2d)}$, however, possesses dependence on dimension, having in the infinite-dimension limit the mean field value $N^{-3/4}$.[11]

Above the second critical point, that is, for $\beta > \tilde{\beta}_c$, the picture is that typical for ferromagnets (without any external perturbations) in the low-temperature regime. The characteristic function is again a ratio of two integrals

$$\left\langle \exp\left(it \sum_{j\in\Omega_n} \varepsilon_j\right)\right\rangle_N = \frac{J_1}{J_2}$$

which are given by

$$J_1 = \int_{\zeta_0-i\infty}^{\zeta_0+i\infty} \frac{1}{\sqrt{\zeta}} \exp\left(2\beta J\zeta m_s^2 - \frac{t^2N^2}{\beta J\pi^2\zeta}\right)$$

$$J_2 = \int_{\zeta_0-i\infty}^{\zeta_0+i\infty} \frac{1}{\sqrt{\zeta}} \exp(\zeta 2\beta J m_s^2)$$

where m_s^2 is given by (5.6) and irrelevant (as $N \to \infty$) terms are omitted [cf. (5.19) and (5.20)]. Consequently,

$$\lim_{N \to \infty} \left\langle \exp\left(itN^{-1} \sum_{j \in \Omega_n} \varepsilon_j \right) \right\rangle_N = \cos\left(\frac{2tm_s\sqrt{2}}{\pi} \right)$$

Thus, for $\beta > \tilde{\beta}_c$ the random variable (magnetization) $N^{-1} \sum_{j \in \Omega_n} \varepsilon_j$ is a dichotomic one taking the values $\pm 2\sqrt{2m_s/\pi}$ with equal probabilities.

One can repeat the above analysis to obtain the characteristic function $X_\gamma(t)$ of the random variable

$$p_\gamma \overset{d}{=} \lim_{n \to \infty} \frac{1}{N^\gamma} \sum_{j \in \Omega_n} \delta_j \varepsilon_j$$

However, calculations in this case are technically more complicated; thus we give a bit more detail than in the cases already considered.

We start from the expression

$$X_\gamma^{(n)}(t) = \frac{1}{\Theta_N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mu_n(d\varepsilon) \exp\left[itN^{-\gamma} \sum_{j \in \Omega_n} \delta_j \varepsilon_j - \beta H_n(\varepsilon) \right]$$

After diagonalization of the quadratic part of the Hamiltonian one obtains

$$X_\gamma^{(n)}(t) = \frac{1}{\Theta_N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mu_n(dy) \exp\left[\beta J \sum_{s \in \Omega_n} \lambda_s y_s^2 + \sum_{s \in \Omega_n} (\beta \alpha_s + itN^{-\gamma} \chi_s) y_s \right]$$

where coefficients χ_s are given by

$$\chi_s = \sum_{j \in \Omega_n} \delta_j V_{j;s} \tag{5.21}$$

Performing the summation in Eq. (5.21) (which needs essentially only patience and some care), one obtains

$$\chi_s = \left(\frac{2}{n+1} \right)^{1/2} n^{(d-1)/2} \prod_{\nu=1}^{d-1} \delta_{s_\nu;1} \begin{cases} 0 & \text{if } s_d \text{ is odd} \\ -\tan \dfrac{\pi s_d}{4(n+1)} & \text{if } s_d \text{ is a multiple of 4} \\ \cot \dfrac{\pi s_d}{4(n+1)} & \text{otherwise} \end{cases}$$

As usual, integration over the variables $\{y_j\}_{j \in \Omega_n}$ can be done after substitution of the integral representation of the delta function into $\mu_n(dy)$; this yields

$$X_\gamma^{(n)}(t)=\left\langle \exp\left[\frac{itN^{-\gamma}}{4J}S_n^{(1)}(z)-\frac{t^2N^{-2\gamma}}{8\beta J}S_n^{(2)}(z)\right]\right\rangle_{z,N} \qquad (5.22)$$

where

$$S_n^{(1)}(z)=\sum_{j\in\Omega_n}\frac{\alpha_j\chi_j}{z-\frac{1}{2}\lambda_j}$$

and

$$S_n^{(2)}(z)=\sum_{j\in\Omega_n}\frac{\chi_j^2}{z-\frac{1}{2}\lambda_j}$$

Using the techniques of Appendix A, one obtains the following expressions for the sums $S_n^{(1)}(z)$ and $S_n^{(2)}(z)$:

$$S_n^{(1)}(z)=2n^{-1}Nh\left[\left(1+\frac{2}{z-d}\right)^{1/2}\frac{x_2^{n+1}(z)+1}{x_2^{n+1}(z)-1}-1-\frac{4}{1-x_1(z)}\frac{x_2^{n/2}(z)}{x_2^{n+1}(z)-1}\right]$$

$$S_n^{(2)}(z)=N(1+n^{-1})\frac{1}{z-d}-n^{-1}N\frac{2}{x_2(z)-x_1(z)}\frac{x_2^{n+1}(z)+1}{x_2^{n+1}(z)-1}$$

$$+n^{-1}N\frac{2}{\tilde{x}_2^2(z)-\tilde{x}_1^2(z)}\sum_{\sigma=\pm 1}\frac{1}{1+\sigma[\frac{1}{2}(z-d)+1]^{1/2}}\frac{\tilde{x}_2^{n+1}(z)+\sigma}{\tilde{x}_2^{n+1}(z)-\sigma}$$

where $\tilde{x}_{1;2}(z)$ are given by

$$\tilde{x}_{1;2}(z)=\frac{1}{\sqrt{2}}\left[(z-d+2)^{1/2}\mp(z-d)^{1/2}\right]$$

The above expressions were derived for the case of even $n/2$. For others values of n similar expressions can be obtained.

In the high-temperature regime, applying the saddle point method for the evaluation of the integral in Eq. (5.22), one obtains

$$X_{1-1/d}(t)\equiv\lim_{N\to\infty}X_{1-1/d}^{(n)}(t)=\exp\left\{\frac{ith}{2J}\left[\left(1+\frac{2}{z^*-d}\right)^{1/2}-1\right]\right\}$$

which, as could have been expected, coincides with (5.11) for $h_1=h_2=h$.

One can calculate the characteristic function

$$\tilde{\chi}_n(t)\equiv\left\langle \exp it\left[\sum_{j\in\Omega_n}\delta_j\varepsilon_j-\frac{1}{4J}S_n^{(1)}(z_n^*(\zeta))\right]\right\rangle_N$$

for fluctuations of p_γ in much the same way as was done for the total spin. In the low-temperature regime (that is, when $z^* > d$) one obtains

$$\lim_{N\to\infty} \tilde{\chi}_n(N^{-1/2}t) = \exp\left(-\frac{t^2}{8\beta J}\frac{1}{z^*-d}\right)$$

As could have been expected, the distribution of fluctuations of $p_{1/2}$ is the same as that for the total spin normalizes by $N^{-1/2}$.

At the first critical point one needs to rescale the integration variable in Eq. (5.22) according to Eq. (3.26) prior to applying the saddle point method. As functions of thed rescaled variable ζ the leading terms of the asymptotic expansions of $S_n^{(1)}(z_n(\zeta))$ and $S_n^{(2)}(z_n(\zeta))$ are given by

$$S_n^{(1)}(z_n(\zeta)) \sim \frac{2Nh}{n}\left(\frac{2}{\zeta}\right)^{1/2} \begin{cases} \sqrt{n} & \text{when} \quad d=3 \\ (n \ln n)^{1/3} & \text{when} \quad d=4 \\ n^{1/3} & \text{when} \quad d \geqslant 5 \end{cases}$$

$$S_n^{(2)}(z_n(\zeta)) \sim \frac{N}{\zeta} \begin{cases} n & \text{when} \quad d=3 \\ (n \ln n)^{2/3} & \text{when} \quad d=4 \\ n^{2/3} & \text{when} \quad d \geqslant 5 \end{cases}$$

For the characteristic function $X(t)$ of the random variable

$$\frac{1}{g_d(n)} \sum_{j\in\Omega_n} \delta_j \varepsilon_j$$

where the normalizing factor $g_d(n)$ is given by Eq. (5.14), application of the saddle point method yields

$$X(t) = \exp\left[it \frac{h}{J(2\zeta^*)^{1/2}}\right]$$

where the saddle point ζ^* is given by Eq. (3.27).

The proper normalization factor for fluctuations of p_γ at the critical point $\beta = \beta_c$ is $[N^{-1}v_d(n)]^{1/2}$, and is the same as for the total spin. For the characteristic function $\tilde{\chi}_n([N^{-1}v_d(n)]^{1/2}t)$ one obtains

$$\lim_{N\to\infty} \tilde{\chi}_n([N^{-1}v_d(n)]^{1/2}t) = \exp\left(-\frac{t^2}{8\beta J\zeta^*}\right)$$

where $v_d(n)$ is given by (3.26). Thus, at the first critical point, fluctuations of the random variable p_γ remain Gaussian, as in the high-temperature regime; however, the normalizing factor becomes significantly different from $N^{-1/2}$, the one relevant to the sum of independent random variables.

For $\beta_c < \beta < \tilde{\beta}_c$ one needs to rescale the integration variable in Eq. (5.22) via $z = d + n^{-2}\zeta$. After the rescaling the leading terms of the asymptotic expansions of $S_n^{(1)}(d+n^{-2}\zeta)$ and $S_n^{(2)}(d+n^{-2}\zeta)$ are given by

$$S_n^{(1)}(d+n^{-2}\zeta) \sim 2Nh\left(\frac{2}{\zeta}\right)^{1/2} \tanh\left[\frac{1}{2}\left(\frac{\zeta}{2}\right)^{1/2}\right] \equiv Ns_1(\zeta)$$

$$S_n^{(2)}(d+n^{-2}\zeta) \sim Nn^2\zeta^{-1}\left\{1 - 2\left(\frac{2}{\zeta}\right)^{1/2} \tanh\left[\frac{1}{2}\left(\frac{\zeta}{2}\right)^{1/2}\right]\right\} \equiv Nn^2 s_2(\zeta)$$

for $\zeta > 0$, and one has to perform an analytic continuation to obtain expressions for $s_{1,2}(\zeta)$ on the interval $-\pi^2/2 < \zeta \leqslant 0$.

Application of the saddle point method yields

$$X_1(t) = \exp\left[\frac{it}{4J} s_1(\zeta^*)\right]$$

where ζ^* is the solution of Eq. (3.22).

Let us consider now the characteristic function $\tilde{\chi}_n(t)$ for fluctuation of p_γ. One needs to introduce for the variable t the rescaling factor $n^{-1}N^{-1/2}$ in order to compensate the growth of $S_n^{(2)}(d+n^{-2}\zeta)$. One can apply then the saddle point method in much the same way as was done for the total spin fluctuations; cf. (5.17). On applying it, one arrives at

$$\lim_{N\to\infty} \tilde{\chi}_n(N^{-1/2-1/d}t) = \exp\left[-\frac{t^2}{8\beta J} s_2(\zeta^*) + t^2 \frac{[s_1'(\zeta^*)]^2}{2\tau''(\zeta^*)}\right]$$

Let us note now that the Taylor expansions of the functions $s_{1,2}(\zeta)$ and $\tau''(\zeta)$ at the point $\zeta = -\frac{1}{2}\pi^2$ are given by

$$s_2(\zeta) = \frac{4h}{\pi} + \frac{2-\pi}{\pi^3}\, 2h\left(\zeta + \frac{1}{2}\pi^2\right) + O\left(\zeta + \frac{1}{2}\pi^2\right)^2$$

$$s_2(\zeta) = \frac{2(4-\pi)}{\pi^3} + \frac{8(3-\pi)}{\pi^5}\left(\zeta + \frac{1}{2}\pi^2\right) + O\left(\zeta + \frac{1}{2}\pi^2\right)^2$$

$$\tau''(\zeta) = \frac{\beta h^2}{2J}\left[\frac{1}{2\pi^2} - \frac{\pi^2-6}{4\pi^4}\left(\zeta + \frac{1}{2}\pi^2\right) + O\left(\zeta + \frac{1}{2}\pi^2\right)^2\right]$$

Hence, after further rescalings of the integration variable, which are necessary both at the second critical point and for $\beta > \tilde{\beta}_c$, the main terms of the asymptotic expansions of $S_n^{(1)}(z_n^*(\tilde{\zeta}))$ and $S_n^{(2)}(z_n^*(\tilde{\zeta}))$ become independent of $\tilde{\zeta}$ (the new integration variable). Consequently, for $\beta \geqslant \tilde{\beta}_c$,

$$X_1(t) = \exp\left(\frac{ith}{\pi J}\right) \tag{5.23}$$

and

$$\lim_{N\to\infty} \tilde{\chi}_n(N^{-1/2-1/d}t)=\exp\left[-\frac{t^2}{4\beta J}\left(\frac{4-\pi}{\pi^2}-\frac{2(2-\pi)^2}{\pi^4}\right)\right]$$

It follows from (5.23) that the mean value of the order parameter p_1 does not depend on the temperature in the thermodynamic limit for $\beta > \tilde{\beta}_c$.

6. DISCUSSION AND CONCLUDING REMARKS

First of all we would like to explain why we considered the internal b.c. (2.2) instead of the most often considered completely periodic b.c., in spite of the fact that the latter are the most convenient from the computational point of view. The answer is very simple: the spherical model with completely periodic internal b.c. possesses unphysical (for a ferromagnet) properties. Indeed, consider, for instance, the spherical model with completely periodic internal b.c. and the external b.c.,

$$h_{i_1,i_2,\ldots,i_d}=\begin{cases} h_1 & \text{if } \; i_d=1 \\ h_2 & \text{if } \; i_d=\dfrac{n}{2}+1 \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

In the completely periodic case these external b.c. are a natural analog of Eq. (2.3). For this model, below the critical temperature $T_c=\beta_c^{-1}$ spins at sites

$$k\in\left\{l\in\Omega_n\colon\, l_d\leqslant n^{1/3},\text{ or } n-l_d\leqslant n^{1/3},\text{ or }\left|\frac{n}{2}-l_d\right|\leqslant n^{1/3}\right\}$$

take abnormally large values of the order $n^{1/3}$, while most of the remaining spins are practically zero. To demonstrate this, let us notice that the eigenvectors and eigenvalues of the interaction matrix $\hat{C}^{(n)}$ are given by

$$\begin{aligned}\tilde{\mathbf{V}}_j=\Bigg\{\tilde{V}_{j;m}=N^{-1/2}\prod_{k=1}^{d}\Bigg[&\cos\frac{2\pi(j_k-1)(m_k-1)}{n}\\ &+\sin\frac{2\pi(j_k-1)(m_k-1)}{n}\Bigg]\Bigg\}_{m\in\Omega_n}\end{aligned} \tag{6.2}$$

$$\tilde{\lambda}_j=2\sum_{k=1}^{d}\cos\frac{2\pi(j_k-1)}{n} \tag{6.3}$$

when the internal b.c. are totally periodic. For the average values of the spin variables, by analogy with (4.2), one obtains

$$\langle \varepsilon_k \rangle_N = \frac{1}{4J} \left\langle \sum_{s \in \Omega_n} \frac{\tilde{V}_{k;s}\tilde{\alpha}_s}{z - \frac{1}{2}\tilde{\lambda}_s} \right\rangle_{z,N}, \qquad k \in \Omega_n \tag{6.4}$$

where

$$\tilde{\alpha}_s = \frac{\sqrt{N}}{n}[h_1 + (-1)^{s_d+1}h_2] \prod_{\nu=1}^{d-1} \delta_{s_\nu;1}$$

and $\langle \cdot \rangle_{z,N}$ is defined by (4.3), where $\Phi_N(z)$ must be modified to take into account (6.2), (6.3). Performing the summation in (6.4), one obtains

$$\tilde{\Sigma}_k(z) = \frac{2}{[x_2(x) - x_1(z)][x_2^n(z) - 1]} \times \begin{cases} h_1(x_2^{k-1}(z) + x_2^{n-k+1}(z)) + h_2(x_2^{n/2+k-1}(z) + x_2^{n/2-k+1}(z)) \\ \qquad \text{if} \quad k = 1, 2,..., n/2 \\ h_1(x_2^{k-1}(z) + x_2^{n-k+1}(z)) + h_2(x_2^{k-1-n/2}(z + x_2^{(3/2)n-k+1}(z)) \\ \qquad \text{if} \quad k = n/2 + 1,..., n \end{cases}$$

For all $\beta > \beta_c$ and h_1, h_2 (we suppose that at least one of h_1, h_2 differs from zero) the natural scale for the integration variable in (6.4) is introduced by the change of variables $z = z_n(\zeta) = d + n^{-2/3}\zeta$ (it can be established by an examination of the field-induced term corresponding to $\tilde{\alpha}_s$).

We are now in a position to analyze the averages $\langle \varepsilon_k \rangle_N$ for k in the strip $k \in A_1 \equiv \{l \in \Omega_n\colon 1 \leqslant l_d \leqslant n/4\}$. The average of the spins from the rest of Ω_n can be analyzed in the same way. Since for large N the main contribution to the integral in Eq. (6.4) comes from a small vicinity of the saddle point ζ^*, one can rewrite $\langle \varepsilon_k \rangle_N$ (for $k \in A_1$), omitting irrelevant terms, as

$$\langle \varepsilon_k \rangle_N \sim \frac{n^{1/3}h_1}{4J(2\zeta^*)^{1/2}} [1 + n^{-1/3}(2\zeta^*)^{1/2}]^{-k_d}$$

Hence, as $N \to \infty$,

$$\langle \varepsilon_k \rangle_N \sim \frac{n^{1/3}h_1}{4J(2\zeta^*)^{1/2}} \begin{cases} 1 & \text{if} \quad k_d = o(n^{1/3}) \\ \exp[-n^{\gamma - 1/3}\alpha(2\zeta^*)^{1/2}] & \text{if} \quad k_d \sim \alpha n^\gamma, \quad \frac{1}{3} \leqslant \gamma < 1 \end{cases}$$

Thus, the completely periodic spherical model with the external b.c. (6.1) possesses an instability similar to that found by Lieb and Thompson[20] and Barber *et al.*[9] The internal b.c. (2.2) create a repulsion from the edges of the hypercube corresponding to $j_d = 1$ and n, preventing the anomalous "condensation." Being evidently unphysical for a ferromagnet, this phenomenon seems to be in agreement with physical intuition if one thinks of the spherical model as describing a (binary) liquid with very weak surface tension. Indeed, if one imagines a liquid in a vessel, in the absence of gravity, then it is reasonable to expect that all of the liquid gathers on the walls of the vessel due to wetting.

A phase diagram of the model (2.1)–(2.5) for $h_1 = -h_2 = h$ is shown in Fig. 4. Qualitatively it is the same as that obtained in ref. 23, where the spherical model with a perturbation by an infinitesimal inhomogeneous external field—in the spirit of the quasiaverages approach—was considered. Phase I is the ordinary paramagnetic phase. Phase II is characterized by a nonzero value of the order parameter p_1 [see (5.1)] and zero value of the magnetization. Distributions of the single spin variables in phase II are Gaussian. The phase transition I → II is characterized by critical indices typical for the ferromagnetic phase transition in the spherical model without any external perturbation. The distribution of the normalized total spin, however, remains Gaussian (for $h \neq 0$), which suggests that the presence of the external b.c. weakens correlations in the model at $\beta = \beta_c$. Phase III is characterized by spontaneous symmetry breaking, since the magnetization takes values $\pm m_s$ with equal probabilities, and the order parameter p_1 is "frozen" at a nonzero value. The distribution densities of

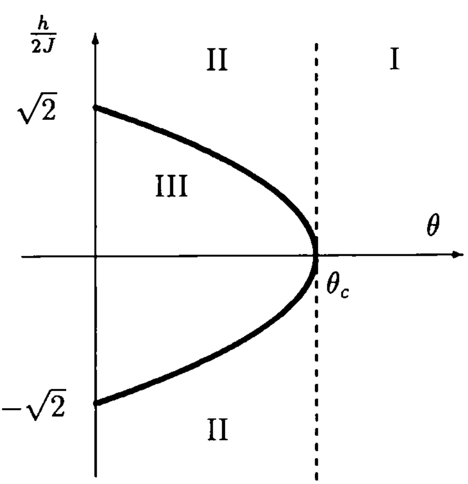

Fig. 4. The phase diagram of the model for $h_1 = -h_2 = h$. Phase I is the ordinary paramagnetic phase, phase II is a phase where the external b.c. prevent "spontaneous symmetry breaking," and in phase III "spontaneous symmetry breaking" takes place, making the phase similar to a ferromagnetic phase.

the single spin variables in this phase are linear combinations of Gaussian densities. The phase transition II → III is characterized by the mean field critical indices. The distribution of the normalized total spin becomes non-Gaussian, which is typical for critical points corresponding to an onset of spontaneous magnetization. The point $\beta = \beta_c$ is called a critical point since it is a point of nonanalyticity of the free energy per site $f(\beta)$. In spite of the fact that $f(\beta)$ is analytic at $\beta = \tilde{\beta}_c$ it is reasonable to call it a critical point also, since $\tilde{\beta}_c$ is a point of nonanalyticity of the macroscopic variables: magnetization and p_1.

As follows from the explicit formula for the single spin distribution [see Eq. (5.7)], $\langle \varepsilon_j^2 \rangle \neq 1$ (generally speaking) in the low-temperature regime. Hence, the behavior of the spherical model and the generalized spherical model[17] (which is the large-n limit of the n-vector models) is different in this regime. It is not excluded that the difference is qualitative, that is, the generalized spherical model possibly does not have the second critical temperature. It would be very interesting to clarify this question.

In conclusion, let us say a few words about limiting Gibbs states for the model. We are in a similar situation to Abraham and Robert[4] in that we did not find any translation-variant (bulk) Gibbs states irrespective of how high the dimensionality is. The existence of translation-variant surface Gibbs states for $\beta < \beta_c$ due to the presence of the external b.c. is not surprising; see the characteristic function of a single spin $\langle \exp(it\varepsilon_k) \rangle_{0;0}$ given by (5.3). What was unexpected (by the author) is that such Gibbs states still exist for $\beta > \beta_c$. Despite the average values of the single spin variables being translationaly invariant (on the microscopic scale, which is the only relevant scale for the Gibbs states) at the surface, their variances are not; see Eqs. (5.4), (5.5).

Concerning the translation-invariant bulk Gibbs states, we would like to stress the existence [for $\frac{1}{2}(h/2J)^2 < 1$ and $\beta > \tilde{\beta}_c$] of the two-parameter family of mixed states with single spin distribution densities [cf. (5.7)],

$$p(\varepsilon) = \frac{1}{2} \left(\frac{\beta}{2\pi\beta_c} \right)^{1/2} \sum_{\sigma = \pm 1} \exp\left[-\frac{\beta}{2\beta_c} (\varepsilon - \kappa - \sigma\mu)^2 \right] \tag{6.5}$$

where $\kappa \in [-h/2J; h/2J]$ and $\mu \in [0; \sqrt{2} m_s]$, with m_s given by (5.6). The existence of this family suggests the existence of the two-parameter family of pure Gibbs states with the single spin distribution

$$q(\varepsilon) = \left(\frac{\beta}{2\pi\beta_c} \right)^{1/2} \exp\left[-\frac{\beta}{2\beta_c} (\varepsilon - \kappa - \mu)^2 \right]$$

with the same range for κ and $\mu \in [-\sqrt{2} m_s; \sqrt{2} m_s]$.

APPENDIX A. A "CONTOUR SUMMATION" TECHNIQUE

It is a common practice to replace sums of the type

$$\frac{1}{m}\sum_{k=0}^{m-1} f\left(\cos\frac{2\pi k}{m}\right) \tag{A.1}$$

by the integrals

$$\frac{1}{2\pi}\int_0^{2\pi} f(\cos\omega)\,d\omega \tag{A.2}$$

when one needs to investigate the former for large m, since

$$\frac{1}{m}\sum_{k=0}^{m-1} f\left(\cos\frac{2\pi k}{m}\right) \to \frac{1}{2\pi}\int_0^{2\pi} f(\cos\omega)\,d\omega$$

as $m\to\infty$ for sufficiently regular functions $f(\cdot)$; see, e.g., refs. 10 and 4. Sometimes this is sufficient for the problem under consideration, such as when Berlin and Kac studied the homogeneous ferromagnetic spherical model. Sometimes, however, one needs to know how fast the sum (A.1) converges to the integral (A.2). Then a technique of one or another type, quite often sophisticated, is used to estimate the rate of convergence; see, e.g., refs. 13 and 8.

It turns out that a large class of sums of the type (A.1) can be calculated exactly. The reason for this simplification is in part the same as makes possible the explicit calculation of the integrals (A.2) for a wide range of analytic functions $f(\cdot)$. As may be found in any textbook on functions of a complex variable, one obtains

$$\frac{1}{2\pi}\int_0^{2\pi} f(\cos\omega)\,d\omega = a_0 \tag{A.3}$$

if the expansion of the function $\tilde{f}(e^{i\omega}) = f(\cos\omega)$ in the Laurent series is given by $\tilde{f}(e^{i\omega}) = \sum_{n=-\infty}^{\infty} a_n e^{in\omega}$. Thus the main ingredients for the validity of Eq. (A.3) are the ability to expand an analytic function in a uniformly convergent Laurent series and the fact that the integral of $e^{in\omega}$ over the interval $[0; 2\pi]$ equals zero unless $n=0$. If one considers the sum (A.1) instead of the integral (A.2), one of course still enjoys the ability to expand the function $\tilde{f}(\cdot)$ in the Laurent series and the following identity for the sums of exponentials is valid:

$$\frac{1}{m}\sum_{k=0}^{m-1} \exp\left(i\frac{2\pi kr}{m}\right) = \begin{cases} 1 & \text{if } r = lm, \quad l\in Z \\ 0 & \text{otherwise} \end{cases} \tag{A.4}$$

Thus, the analog of (A.3) is

$$\frac{1}{m}\sum_{k=0}^{m-1}{}' f\left(\cos\frac{2\pi k}{m}\right)=\sum_{l=-\infty}^{\infty} a_{lm} \tag{A.5}$$

If one can sum up the infinite series in (A.5), one obtains an explicit expression for the sum under consideration The point is that one can do this for a large class of functions $f(\cdot)$ which includes all rational functions and their logarithms.

We consider a somewhat more general class of sums than (A.1), namely

$$\frac{1}{m}\sum_{k=0}^{m-1}{}' f\left(\cos\frac{\pi(2k+\lambda)}{m}\right)\equiv\frac{1}{m}\sum_{k=0}^{m-1}{}' \tilde{f}\left(\exp\frac{i\pi(2k+\lambda)}{m}\right) \tag{A.6}$$

where λ is an arbitrary real number. Suppose that $\tilde{f}(z)=P_1(z)/P_2(z)$, where $P_i(z)$, $i=1, 2$, are two polynomials. Then one can expand $1/P_2(z)$ in a sum of simple fractions, so that

$$\tilde{f}(z)=\sum_{r=1}^{v}\sum_{q=-l}^{l}\frac{b_q c^r z^q}{(z-x_r)^{s_r}}=\sum_{r=1}^{v}\sum_{q=-l}^{l}\frac{b_q c_r z^q}{(s^r-1)!}\frac{d^{s_r-1}}{dx_r^{s_r-1}}\frac{1}{z-x_r} \tag{A.7}$$

(we require, for the sake of less ambiguity, that all $x_r\neq 0$). This in fact solves the problem of summation in (A.6), since

$$\sigma_{m;q;\lambda}(x)\equiv\frac{1}{m}\sum_{k=0}^{m-1}{}'\frac{\exp[i\pi(2k+\lambda)q/m]}{x-\exp[i\pi(2k+\lambda)/m]}$$

$$=\frac{\exp(i\pi\lambda f_0)\,x^{-1-f_0m+q}}{1-\exp(i\pi\lambda)\,x^{-m}} \tag{A.8}$$

where f_0 is an integer satisfying $(f_0-1)m<q\leqslant f_0 m$. From (A.6)–(A.8) one concludes that

$$\frac{1}{m}\sum_{k=0}^{m-1}{}' f\left(\cos\frac{\pi(2k+\lambda)}{m}\right)=-\sum_{r=1}^{v}\sum_{q=-l}^{l}\frac{b_q c_r}{(s_r-1)!}\frac{d^{s_r-1}}{dx_r^{s_r-1}}\sigma_{m;q;\lambda}(x_r) \tag{A.9}$$

for any m.

To obtain (A.8) let us suppose for a moment that $|x|>1$; then, using the formula for the sum of a geometrical series, we obtain

$$\sigma_{m;q;\lambda}(x) = \sum_{s=0}^{\infty} \frac{1}{mx^{s+1}} \sum_{k=0}^{m-1} \exp\left[i\frac{\pi(2k+\lambda)}{m}(q+s) \right]$$
$$= \sum_{s=0}^{\infty} \frac{1}{mx^{s+1}} \begin{cases} me^{i\pi f\lambda} & \text{if } \quad q+s=fm, \quad f\in Z \\ 0 & \text{otherwise} \end{cases}$$
$$= \frac{e^{i\pi\lambda f_0} x^{-1-f_0 m+q}}{1-e^{i\pi\lambda}x^{-m}} \tag{A.10}$$

Since $\sigma_{m;q;\lambda}(x)$ and the final expression in Eq. (A.10) are analytic functions in the whole complex plane of the variable x except for a few poles, the analytic continuation from $|x|>1$ yields (A.8) for all points of analyticity.

The situation in the case $\tilde{f}(z) \equiv \ln[P_1(z)/P_2(z)]$ is even simpler than for a rational function. After factorization of the polynomials one has

$$\tilde{f}(z) = \sum_{r=1}^{l} s_r \ln(1-x_r z) + c \tag{A.11}$$

where $\ln(\cdot)$ is the principal branch of the logarithm, i.e., there is a branch cut along the negative semiaxes and $\ln(1)=0$. Thus, it is sufficient to calculate the sum

$$\delta_m(x) = \frac{1}{m} \sum_{k=0}^{m-1} \ln\left[1 - x\exp\left(i\frac{\pi(2k+\lambda)}{m} \right) \right] \tag{A.12}$$

Suppose, again, for a moment that $|x|<1$; then expanding the logarithm in the Taylor series and using Eq. (A.4), one obtains

$$\delta_m(x) = -\frac{1}{m} \sum_{k=1}^{m} \sum_{q=1}^{\infty} \frac{x^q \exp[i\pi(2k+\lambda)q/m]}{q}$$
$$= -\sum_{f=1}^{\infty} \frac{(x^m)^f \exp(i\pi\lambda f)}{fm} = m^{-1}\ln\{1-\exp(i\pi\lambda)x^m\} \tag{A.13}$$

Since $\delta_m(x)$ given by (A.12) and $m^{-1}\ln(1-e^{i\pi\lambda}x^m)$ are analytic functions throughout the complex plane (of the variable x) except for a branch cut, the analytic continuation allows one to conclude that Eq. (A.13) is valid at all points of analyticity. Thus in the case

$$\tilde{f}(z) = \ln\frac{P_1(z)}{P_2(z)}$$

one obtains

$$\frac{1}{m}\sum_{k=0}^{m-1} f\left(\cos\frac{\pi(2k+\lambda)}{m} \right) = c + m^{-1} \sum_{r=1}^{l} s_r \ln(1-e^{i\pi\lambda}x_r^m) \tag{A.14}$$

Let us now demonstrate the application of formulas (A.9) and (A.14). According to Eq. (3.8), the field-induced term contains the sum

$$t_n(z) = \sum_{s=1}^{n} \frac{[h_1 + (-1)^{s+1} h_2]^2 \sin^2[\pi s/(n+1)]}{z - d + 1 - \cos[\pi s/(n+1)]} \tag{A.15}$$

Using elementary transformations, one can rewrite this as

$$t_n(z) = \frac{(h_1 + h_2)^2}{4} \sum_{s=0}^{n} \frac{1 - \cos[2\pi(2s+1)/(n+1)]}{z - d + 1 - \cos[\pi(2s+1)/(n+1)]} + \frac{(h_1 - h_2)^2}{4} \sum_{s=0}^{n} \frac{1 - \cos[4\pi s/(n+1)]}{z - d + 1 - \cos[2\pi s/(n+1)]} \tag{A.16}$$

Comparing Eq. (A.16) with Eq. (A.9), one sees that the first (second) sum in Eq. (A.16) corresponds to $\lambda = 1$ ($\lambda = 0$), $m = n + 1$ in Eq. (A.9). Noting the identity

$$\frac{1}{z - d + 1 - \cos\phi} = \frac{2}{x_2(z) - x_1(z)} \left[\frac{x_2(z)}{x_2(z) - e^{i\phi}} + \frac{x_1(z)}{e^{i\phi} - x_1(z)} \right] \tag{A.17}$$

where $x_{1,2}(z)$ is given by (3.10), and comparing the sums in Eq. (A.16) with Eq. (A.7), one obtains the following values for the parameters in Eq. (A.9): $v = l = 2$; $s_{1,2} = 1$; $x_{1,2} = x_{1,2}(z)$; $c_{1,2} = 2x_{1,2}/(x_{2,1} - x_{1,2})$; $b_{\pm 2} = -\frac{1}{2}$, $b_{\pm 1} = 0$, $b_0 = 1$. Thus after some algebra Eq. (A.9) yields

$$t_n(z) = \frac{(n+1)\, x_2(z)}{2} \left[(h_1 + h_2)^2 \frac{x_2^{n-1}(z) + 1}{x_2^{n+1}(z) + 1} + (h_1 - h_2)^2 \frac{x_2^{n-1}(z) - 1}{x_2^{n+1}(z) - 1} \right]$$

which yields Eq. (3.9) for the field-induced term. All the other summations in the paper involving rational functions $f(\cdot)$ can be carried out similarly.

As an application of the formula (A.14), let us calculate the sum

$$\sum_{j=1}^{n} \ln \left[z - \cos \frac{2\pi}{n} (j-1) \right]$$

Comparing this expression with Eq. (A.11), one identifies with the help of the identity

$$z - \cos\phi = \tfrac{1}{2} x_+(z) [1 - x_-(z) e^{-i\phi}][1 - x_-(z) e^{i\phi}]$$

where $x_\pm(z) = z \pm (z^2 - 1)^{1/2}$, the values of the coefficients in Eq. (A.11): $l = 2$, $\lambda = 0$, $c = \ln \frac{1}{2} x_+(z)$, $s_{1,2} = 1$, $x_{1,2} = x_-(z)$. Hence, Eq. (A.14) yields

$$\frac{1}{n} \sum_{j=1}^{n} \ln \left[z - \cos \frac{2\pi}{n} (j-1) \right] = \ln \frac{1}{2} x_+(z) + 2n^{-1} \ln[1 - x_-^n(z)] \tag{A.18}$$

APPENDIX B. ASYMPTOTIC BEHAVIOR OF SOME SEQUENCES OF SUMS

In this appendix we study the rate of convergence of the sequences of sums $L_d^{(n)}(z)$ and $W_d^{(n)}(z)$, given by Eqs. (3.6) and (3.13), respectively, toward their limits. Although these problems have been studied earlier (see, e.g., refs. 8 and 13), we would like to present a derivation of the asymptotic formulas for the sums to demonstrate the application of the summation technique described in Appendix A. In doing this we will be able to refine previously known results and to derive them, in our opinion, with considerably less labor than in refs. 8 and 13.

We study first the sequence

$$W_d^{(n)}(z) = \frac{1}{N} \sum_{j \in \Omega_n} \frac{1}{z - \frac{1}{2}\lambda_j} \tag{B.1}$$

where $\lambda_j \equiv \lambda_{j_1,\dots,j_d}$ is given by (2.8). One can perform the summation over j_1 in (B.1) by using the identity

$$\sum_{j_1=1}^{n} \frac{1}{z - \cos[2\pi(j_1 - 1)/n]} = \frac{2n}{x_+(z) - x_-(z)} \left[1 + \frac{2}{x_+^n(z) - 1} \right] \tag{B.2}$$

where $x_\pm(z) = z \pm (z^2 - 1)^{1/2}$, which can be obtained using the technique of Appendix A. Summation over j_1 yields

$$W_d^{(n)}(z) = \frac{n}{N} \sum_{j_2,\dots,j_d} \frac{2}{x_+(z_1) - x_-(z_1)} + \frac{n}{N} \Delta_1^{(d)}(z) \tag{B.3}$$

where

$$\Delta_1^{(d)}(z) = \sum_{j_2,\dots,j_d} \frac{4}{x_+(z_1) - x_-(z_1)} \frac{1}{x_+^n(z_1) - 1}$$

and

$$z_1 = z - \sum_{k=2}^{d-1} \cos \frac{2\pi(j_k - 1)}{n} - \cos \frac{\pi j_d}{n+1} \tag{B.4}$$

Now it is convenient to rewrite (B.3) using the ["converse" to (B.2)] identity

$$\frac{2}{x_+(z) - x_-(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\omega}{z - \cos \omega}$$

which can be obtained for $z > 1$ from (B.2) if one divides by n and passes to the limit $n \to \infty$. This yields

$$W_d^{(n)}(z) = \frac{n}{N} \sum_{j_2,\ldots,j_d} \frac{1}{2\pi} \int_0^{2\pi} \frac{d\omega_1}{z_1 - \cos\omega_1} + \frac{n}{N} \Delta_1^{(d)}(z)$$

and now one can perform the summation over j_2 under the sign of the integral. Performing summations over $j_2, \ldots, j_{d-1}$, one obtains

$$W_d^{(n)}(z) = \frac{n^{d-1}}{N} \sum_{j_d=1} \frac{1}{(2\pi)^{d-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d\omega_1 \cdots d\omega_{d-1}}{z - \sum_{k=1}^{d-1} \cos\omega_k - \cos[\pi j_d/(n+1)]} + \frac{1}{N} \sum_{k=1}^{d-1} n^k \Delta_k^{(d)}(z)$$

where for $k = 1, \ldots, d-1$

$$\Delta_k^{(d)}(z) = \sum_{j_{k+1},\ldots,j_d} \frac{1}{(2\pi)^{k-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d\omega_1 \cdots d\omega_{k-1}}{x_+^n(z_k) - 1} \frac{2}{(z_k^2 - 1)^{1/2}}$$

and

$$z_k = z - \sum_{v=1}^{k-1} \cos\omega_v - \sum_{v=k+1}^{d-1} \cos\frac{2\pi}{n}(j_v - 1) - \cos\frac{\pi j_d}{n+1} \qquad \text{(B.5)}$$

The remaining summation over j_d can be performed using the identity

$$\sum_{j=1}^{n} \frac{1}{z - \cos[\pi j/(n+1)]} = \frac{2(n+1)}{x_+(z) - x_-(z)} \left[1 + \frac{2}{x_+^{2(n+1)}(z) - 1}\right] - \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1}\right)$$

which can also be obtained using the technique of Appendix A. On summing, one obtains

$$W_d^{(n)}(z) = W_d(z) - \tfrac{1}{2} n^{-1} [W_{d-1}(z-1) + W_{d-1}(z+1) - 2W_d(z)] + N^{-1} \sum_{k=1}^{d} n^k \Delta_k^{(d)}(z) \qquad \text{(B.6)}$$

where

$$\Delta_d^{(d)}(z) = \left(1 + \frac{1}{n}\right) \frac{1}{(2\pi)^{d-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d\omega_1 \cdots d\omega_{d-1}}{x_+^{2(n+1)}(z_d) - 1} \frac{2}{(z_d^2 - 1)^{1/2}}$$

and

$$z_d = z - \sum_{\nu=1}^{d-1} \cos \omega_\nu$$

Our strategy now is to show that the term $N^{-1} \sum_{k=1}^{d} n^k \Delta_k^{(d)}(z)$ is small, and making use of Eq. (B.6), obtain the asymptotic expansion for $W_d^{(n)}(z)$ from that for $W_d(z)$ which can be found in the paper by Barber and Fisher.[8] For our purposes the following will be sufficient:

$$W_d(z) = W_d(d) + \begin{cases} -\dfrac{1}{\pi}\left(\dfrac{z-d}{2}\right)^{1/2} + O(z-d) & \text{if} \quad d=3 \\ \dfrac{(z-d)\ln(z-d)}{4\pi^2} + O(z-d) & \text{if} \quad d=4 \\ (z-d)\, W_d'(d) + O(z-d)^{3/2} & \text{if} \quad d \geqslant 5 \end{cases} \tag{B.7}$$

Note that the Watson function $W_d(z)$ which we use differs slightly from the one $W_d^{bf}(z)$ used by Barber and Fisher, but coincides with that of Berlin and Kac.[10] The relation between these two functions is $W_d(z) = 2W_d^{bf}(2z)$.

If z is fixed (does not scale with n) and $z > d$ one has $x_+(z_k) \geqslant x_+(z-d+1) > 1$ and hence $\Delta_k^{(d)}(z) = O(\exp[-nc(z)])$ with $c(z)$ strictly positive (for $z > d$). Thus for any given $z > d$ the contribution produced by the term $\sum_{k=1}^{d} n^k \Delta_k^{(d)}(z)$ in the asymptotic expansion of $W_d^{(n)}$ is weaker than any power of n^{-1} and can be neglected.

If z scales with n as $z = z_n(\zeta) = d + n^{-\rho}\zeta$ with $0 < \rho < 2$ the same conclusion is true, since for this scaling of z one has $x''_+(z_n(\zeta)) = O(\exp[n^{1-\rho/2}(2\zeta)^{1/2}])$ and hence

$$\Delta_\nu^{(d)}(d + \zeta n^{-\rho}) = O(\exp[-n^{1-\rho/2} c_1(\zeta)]) \tag{B.8}$$

with $c_1(\zeta)$ strictly positive and increasing for $\zeta > 0$. Thus the asymptotic expansion of $W_d^{(n)}(d + \zeta n^{-\rho})$ follows from Eqs. (B.6) and (B.7).

Now suppose $\rho = 2$, $z = z_n(\zeta) = d + n^{-2}\zeta$. Note that we are interested in the asymptotic expansion of $W_d^{(n)}(z)$ for $z > \lambda_{1,\dots,1} = d - 1 + \cos[\pi/(n+1)]$ and hence for $\rho = 2$ the relevant values of ζ are those in the range $(-\frac{1}{2}\pi^2; \infty)$. For $\zeta > 0$ one can obtain the following estimates:

$$\Delta_k^{(d)}(d + \zeta n^{-2}) = O(n^{-k+2}), \qquad k = 1, \dots, d$$

and hence

$$N^{-1} \sum_{k=1}^{d} n^k \Delta_k^{(d)}(z) = O(n^{-d+2}) \tag{B.9}$$

Taking into account Eqs. (B.6), (B.7), and (B.9), one arrives at

$$W_d^{(n)}(d+n^{-2}\zeta) = W_d(d) + O(n^{-1}) \tag{B.10}$$

for any given $\zeta > 0$ and $d \geqslant 3$.

If $\zeta < 0$ the function $W_d(d+\zeta n^{-2})$ is not well defined and we cannot use the above technique. Nevertheless, to obtain the asymptotic expansion for $\zeta < 0$ let us rewrite $W_d^{(n)}(d+n^{-2}\zeta)$ as

$$W_d^{(n)}(d+n^{-2}\zeta) = w_n(\zeta;\zeta_0) + W_d^{(n)}(d+n^{-2}\zeta_0)$$

where

$$w_n(\zeta;\zeta_0) = W_d^{(n)}(d+n^{-2}\zeta) - W_d^{(n)}(d+n^{-2}\zeta_0)$$

and $\zeta_0 > 0$. Using the Taylor formula, one obtains

$$|w_n(\zeta;\zeta_0)| \leqslant \frac{\zeta_0-\zeta}{n^2} W_d^{(n)\prime}(d+n^{-2}\zeta)$$

The sum

$$W_d^{(n)\prime}(d+n^{-2}\zeta) = N^{-1} \sum_{j\in\Omega_n} \frac{1}{(d+n^{-2}\zeta-\frac{1}{2}\lambda_j)^2}$$

behave as

$$W_d^{(n)\prime}(d+n^{-2}\zeta) = \begin{cases} O(n) & \text{if} \quad d=3 \\ O(\ln n) & \text{if} \quad d=4 \\ O(1) & \text{if} \quad d \geqslant 5 \end{cases} \tag{B.11}$$

Hence, using the expansion (B.10) for $W_d^{(n)}(d+n^{-2}\zeta_0)$, one obtains

$$W_d^{(n)}(d+n^{-2}\zeta) = W_d(d) + \begin{cases} O(n^{-1}) & \text{if} \quad d=3 \\ O(n^{-2}\ln n) & \text{if} \quad d=4 \\ O(n^{-2}) & \text{if} \quad d \geqslant 5 \end{cases} \tag{B.12}$$

When z scales as $z = z_n(\zeta) = d - \frac{1}{2}\pi^2 n^{-2} + \zeta n^{-\rho}$ and $\rho > 2$ we can proceed as follows. In the scale $z_n(\zeta) = d + \zeta n^{-2}$ the contribution produced by the term of $W_d^{(n)}(z)$ corresponding to the largest eigenvalue $\lambda_{1,\dots,1}$ is of $O(n^2 N^{-1})$. Hence, the expansion (B.12) is valid for the sequence of sums

$$\tilde{W}_d^{(n)}(d+n^{-2}\zeta) \equiv \sum_{j\in\Omega_n\setminus\{1,\dots,1\}} \frac{1}{d+n^{-2}\zeta-\lambda_j}$$

as well (in this paper we consider only the case $d \geq 3$). Since in the scales $z_n(\zeta) = d - \frac{1}{2}\pi^2 n^{-2} + \zeta n^{-\rho}$, $\rho > 2$, the distance from $z_n(\zeta)$ to the next largest eigenvalue is still of the order n^{-2}, Eq. (B.12) is still valid for $\tilde{W}_d^{(n)}(z_n(\zeta))$ in those scales. Thus

$$\tilde{W}_d^{(n)}(d - \tfrac{1}{2}\pi^2 n^{-2} + \zeta n^{-\rho}) = W_d(d) + O(n^{-1}) \tag{B.13}$$

for all $\rho > 2$.

We can study the sequence

$$L_d^{(n)}(z) = \frac{1}{N} \sum_{j \in \Omega_n} \ln\left(z - \frac{1}{2}\lambda_j\right) \tag{B.14}$$

where λ_j is given by (2.8), in much the same way as we did for $W_d^{(n)}(z)$. One can perform the summation over j_1 in (B.14) using the identity (A.18). Using next the "converse" identity

$$\frac{1}{2\pi}\int_0^{2\pi} d\omega \ln(z - \cos\omega) = \ln\frac{1}{2}x_+(z)$$

which is the limiting ($n \to \infty$) form of (A.18) for $z > 1$, one obtains

$$L_d^{(n)}(z) = \frac{n}{N}\sum_{j_2;\dots;j_d} \frac{1}{2\pi}\int_0^{2\pi} d\omega_1 \ln(z_1 - \cos\omega_1) + \frac{2}{N}\sum_{j_2;\dots;j_d} \ln[1 - x_-^n(z_1)]$$

where z_1 is given by Eq. (B.4). Performing the summation over $j_2,\dots, j_{d-1}$ along similar lines, one arrives at

$$L_d^{(n)}(z) = \frac{1}{n}\sum_{j_d=1}^{n} L_{d-1}\left(z - \cos\frac{\pi j_d}{n+1}\right) + \frac{1}{N}\sum_{\nu=1}^{d-1} n^{\nu-1}\mathcal{D}_\nu^{(d)}(z) \tag{B.15}$$

where $L_d(z)$ is given by (3.11),

$$\mathcal{D}_\nu^{(d)}(z) = \frac{2}{(2\pi)^{\nu-1}}\int_0^{2\pi}\cdots\int_0^{2\pi} d\omega_1\cdots d\omega_{\nu-1} \sum_{j_{\nu+1};\dots;j_d} \ln[1 - x_-^n(z_\nu)]$$

and (for $\nu = 1,\dots, d-1$), z_ν is given by Eq. (B.5). The remaining summation over j_d can be performed [by switching the orders of summation and of integration in the integral representation (3.11) for $L_d(z)$] using the identity

$$\sum_{j_d=1}^{n} \ln\left(z - \cos\frac{\pi j_d}{n+1}\right)$$
$$= (n+1)\ln\frac{1}{2}x_+(z) + \ln[1 - x_-^{2(n+1)}(z)] - \frac{1}{2}\ln(z^2 - 1)$$

which can be derived in much the same way as Eq. (A.18). On summing, one obtains

$$L_d^{(n)}(z)=\left(1+\frac{1}{n}\right)L_d(z)-\frac{1}{2n}\left[L_{d-1}(z-1)+L_{d-1}(z+1)\right]$$
$$+\frac{1}{N}\sum_{\nu=1}^{d} n^{\nu-1}\mathscr{D}_\nu^{(d)}(z) \tag{B.16}$$

where $z_d=z-\sum_{k=1}^{d}\cos\omega_k$ and

$$\mathscr{D}_d^{(d)}(z)=\frac{1}{(2\pi)^{d-1}}\int_0^{2\pi}\cdots\int_0^{2\pi}d\omega_1\cdots d\omega_{d-1}\ln[1-x_-^{2(n+1)}(z_d)]$$

We are now going to estimate $\mathscr{D}_\nu^{(d)}(z)$ and to obtain the desirable asymptotic expansion from Eq. (B.16) using the asymptotic expansion for $L_d(z)$ given by Eq. (3.25) much as we have done for $W_d^{(n)}(z)$.

We obtain first an estimate for $\mathscr{D}_\nu^{(d)}(z)$ when z is independent of n and $z>d$. Obviously $z_\nu\geqslant 1+z-d$ and

$$x_-(z_\nu)\leqslant x_-(1+z-d)<1 \tag{B.17}$$

Hence

$$|\mathscr{D}_\nu^{(d)}(z)|\leqslant 2n^{d-\nu}|\ln[1-x_-^n(1+z-d)]| \tag{B.18}$$

which means that all $\mathscr{D}_\nu^{(d)}(z)$ become "exponentially" small for $z>d$ as $n\to\infty$. Thus,

$$L_d^{(n)}(z)=L_d(z)-\tfrac{1}{2}n^{-1}[L_{d-1}(z-1)+L_{d-1}(z+1)-2L_d(z)]$$
$$+O(\exp[-\gamma(z)n]) \tag{B.19}$$

where $\gamma(z)$ is strictly positive and increasing for $z>d$.

When z scales as $z=z_n(\zeta)=d+\zeta n^{-\rho}$ the estimate (B.18) is still valid and is quite satisfactory for our purposes if $\rho<2$, since in this case it yields

$$|\mathscr{D}_\nu^{(d)}(d+\zeta n^{-\rho})|=O(n^{d-\nu}\exp[-n^{1-\rho/2}\gamma_1(\zeta)]) \tag{B.20}$$

with $\gamma_1(\zeta)$ strictly positive and increasing for $\zeta>0$. Thus, when $\rho<2$

$$L_d^{(n)}(d+\zeta n^{-\rho})=L_d(d+\zeta n^{-\rho})-\tfrac{1}{2}n^{-1}[L_{d-1}(d-1+\zeta n^{-\rho})$$
$$+L_{d-1}(d+1+\zeta n^{-\rho})-2L_d(d+\zeta n^{-\rho})]$$
$$+O(\exp[-\gamma_1(\zeta)n^{1-\rho/2}])$$

and the desirable expansion can be obtained making use of Eq. (3.25).

Now suppose $\rho = 2$. The estimate

$$\mathscr{D}_k^{(d)}(d+\zeta n^{-2}) = O(n^{-k+1})$$

can be obtained for $\zeta > 0$ as in the case of $\Delta_k^{(d)}(z)$, which yields

$$L_d^{(n)}(d+n^{-2}\zeta) = L_d(d+n^{-2}\zeta) - \tfrac{1}{2}n^{-1}[L_{d-1}(d-1+n^{-2}\zeta) + L_{d-1}(d+1+n^{-2}\zeta) - 2L_d(d+n^{-2}\zeta)] + O(N^{-1})$$

Making use of Eq. (3.25), one obtains

$$L_d^{(n)}(d+n^{-2}\zeta) = L_d(d) + \tfrac{1}{2}n^{-1}[2L_d(d) - L_{d-1}(d-1) - L_{d-1}(d+1)] + \zeta n^{-2}W_d(d) + O(n^{-3}) \qquad \text{(B.21)}$$

Using the same trick as we used for $W_d^{(n)}(z)$, we arrive at the conclusion that Eq. (B.21) is actually true for all $\zeta \in (-\tfrac{1}{2}\pi^2; \infty)$.

Finally, in the scales $z = z_n(\zeta) = d - \tfrac{1}{2}\pi^2 n^{-2} + \zeta n^{-\rho}$, $\rho > 2$, we are interested in the asymptotic behavior of the sequence of sums

$$\tilde{L}_d^{(n)}(z_n(\zeta)) \equiv \sum_{j \in \Omega_n \setminus (1,\dots,1)} \ln(z_n - \tfrac{1}{2}\lambda_j)$$

Note that the sums $L_d^{(n)}(z)$ have a branch point at $z_n = \tfrac{1}{2}\lambda_{1,\dots,1}$, but the sums $\tilde{L}_d^{(n)}(z)$ are analytic till $\tfrac{1}{2}\lambda_{1,\dots,1,2} \approx d - 2\pi^2 n^{-2}$. Since the distance from $d - \tfrac{1}{2}\pi n^{-2}$ to the next maximal eigenvalue is of $O(n^{-2})$ we are still in the scale $z_n = d + \zeta n^{-2}$, $\zeta > -2\pi^2$, when we consider $\tilde{L}_d^{(n)}(z)$ to be at the point $d - \tfrac{1}{2}\pi^2 n^{-2}$. Hence, the asymptotic expansions for $\tilde{L}_d^{(n)}(d - \zeta n^{-2})$ and $\tilde{W}_d^{(n)}(d - \zeta n^{-2})$ as well as an estimate for $\tilde{W}_d^{(n)\prime}(d - \zeta n^{-2})$ [similar to (B.11)] can be established as above for their counterparts without tilde. Using then the Taylor formula

$$\tilde{L}_d^{(n)}(d - \tfrac{1}{2}\pi^2 n^{-2} + \zeta n^{-\rho}) = \tilde{L}_d^{(n)}(d - \tfrac{1}{2}\pi^2 n^{-2}) + \zeta n^{-\rho}\tilde{W}_d^{(n)}(d - \tfrac{1}{2}\pi^2 n^{-2}) + \tfrac{1}{2}\zeta^2 n^{-2\rho}\tilde{W}_d^{(n)\prime}(d - \tfrac{1}{2}\pi^2 n^{-2} + \zeta_1 n^{-\rho})$$

one arrives at

$$\tilde{L}_d^{(n)}(d - \tfrac{1}{2}\pi^2 n^{-2} + \zeta n^{-\rho}) = \tilde{L}_d^{(n)}(d - \tfrac{1}{2}\pi^2 n^{-2}) + \zeta n^{-\rho} W_d^{(n)}(d) + o(n^{-\rho}) \qquad \text{(B.22)}$$

and

$$\tilde{L}_d^{(n)}(d - \tfrac{1}{2}\pi^2 n^{-2}) = L_d(d) + \tfrac{1}{2}n^{-1}[2L_d(d) - L_{d-1}(d-1) - L_{d-1}(d+1)] - \tfrac{1}{2}\pi^2 n^{-2} W_d(d) + o(n^{-2}) \qquad \text{(B.23)}$$

ACKNOWLEDGMENTS

It is a pleasure to acknowledge Denjoe O'Connor for encouraging long-standing interest in this work, reading of the manuscript, and numerous useful comments.

REFERENCES

1. D. B. Abraham, Solvable model with a roughening transition for a planar Ising ferromagnet, *Phys. Rev. Lett.* **44**:1165–1168 (1980).
2. D. B. Abraham, Surface structures and phase transitions—Exact results, in *Phase Transitions and Critical Phenomena*, Vol. 10, C. Domb and J. L. Lebowitz, eds. (Academic Press, New York, 1986), Chapter 1.
3. D. B. Abraham and P. Reed, Interface profile of the Ising ferromagnet in two dimensions, *Commun. Math. Phys.* **49**:35–46 (1976).
4. D. B. Abraham and M. A. Robert, Phase separation in the spherical model, *J. Phys. A: Math. Gen.* **13**:2229–2245 (1980).
5. D. B. Abraham and M. E. Issigoni, Phase separation at the surface of the Ising ferromagnet, *J. Phys. A: Math. Gen.* **13**:L89–L91 (1980).
6. H. Au-Yang, Thermodynamics of an anisotropic boundary of a two-dimensional Ising model, *J. Math. Phys.* **14**:937–946 (1973).
7. R. Z. Bariev, Correlation functions of the semi-infinite two-dimensional Ising model I. Local magnetization, *Theor. Math. Phys.* **40**:623–636 (1979).
8. M. N. Barber and M. E. Fisher, Critical phenomena in systems of finite thickness I. The spherical model, *Ann. Phys.* **77**:1–78 (1973).
9. M. N. Barber, D. Jasnow, S. Singh, and R. A. Weiner, Critical behavior of the spherical model with enhanced surface exchange, *J. Phys. C: Solid State Phys.* **7**:3491–3504 (1974).
10. T. H. Berlin and M. Kac, The spherical model of a ferromagnet, *Phys. Rev.* **86**:821–835 (1952).
11. R. S. Ellis and C. M. Newman, The statistics of the Curie–Weiss models, *J. Stat. Phys.* **19**:149–161 (1978); R. S. Ellis, *Entropy, Large Deviations, and Statistical Mechanics* (Springer-Verlag, Berlin, 1985).
12. M. E. Fisher and A. E. Ferdinand, Interfacial, boundary, and size effects at critical points, *Phys. Rev. Lett.* **19**:169 (1967).
13. M. E. Fisher and V. Privman, First-order transition in spherical models: Finite size scaling, *Commun. Math. Phys.* **103**:527–548 (1986).
14. J. Fröhlich and C.-E. Pfister, Semi-infinite Ising model. I. Thermodynamic functions and phase diagram in absence of magnetic field, *Commun. Math. Phys.* **109**:493–523 (1987); Semi-infinite Ising model. II. The wetting and layering transition, *Commun. Math. Phys.* **112**:51–74 (1987).
15. A. Isihara, Effect of defects of spin interaction in a simple cubic lattice, *Phys. Rev.* **108**:619–629 (1957).
16. J. S. Joyce, Critical properties of the spherical model, in *Phase Transition and Critical Phenomena*, Vol. 2, C. Domb and M. S. Green, eds. (Academic Press, New York, 1972).
17. H. J. F. Knops, Infinite spin dimensionality limit for nontranslationally invariant interactions, *J. Math. Phys.* **14**:1918–1920 (1973).
18. J. S. Langer, A modified spherical model of a first-order phase transition, *Phys. Rev.* **137**:A1531–A1547 (1965).

19. M. Lax, Relation between canonical and microcanonical ensembles, *Phys. Rev.* **97**:1419–1420 (1955).
20. E. H. Lieb and C. J. Thompson, Phase transition in zero dimensions: A remark on the spherical model, *J. Math. Phys.* **10**:1403–1406 (1969).
21. B. McCoy and T. T. Wu, Theory of Toeplitz determinants and the spin correlations of the two-dimensional Ising model. IV, *Phys. Rev.* **162**:436–475 (1967).
22. S. A. Molchanov and Ju. N. Sudarev, Gibbs states in the spherical model, *Sov. Math. Dokl.* **16**:1254–1257 (1975).
23. A. Patrick, On phase separation in the spherical model of a ferromagnet: Quasiaverage approach, *J. Stat. Phys.* **72**:665–701 (1993).
24. S. Singh, D. Jasnow, and M. N. Barber, Critical behavior of the spherical model with enhanced surface exchange: Two spherical fields, *J. Phys. C: Solid State Phys.* **8**:3408–3414 (1975).
25. C. C. Yan and G. H. Wannier, Observation on the spherical model of a ferromagnet, *J. Math. Phys.* **6**:1833–1838 (1965).